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Short Bearing Analysis Applied to Rotor Dynamics

Part I: Theory

A derivation of the Reynolds equation is presented in fixed Cartesian coordinates which allows rapid solution of the fluid-film bearing forces using the short bearing approximation. The solution for the fluid-film pressure profile is given in terms of the instantaneous position and velocities expressed in the fixed reference frame. The effects of cavitation are approximated by deleting subambient pressures when integrating the pressure profile. The equations for the instantaneous whirl rate and curvature are presented and discussed in relation to journal bearing behavior. Three-dimensional plots of the pressure profile for selected dynamic conditions give a vivid picture of the fluid pressure field. Dimensionless plots of the stiffness and damping coefficients for the short bearing are presented. The results of a linearized stability analysis are presented and compared to other published results. Results of transient response analysis will be presented in Part 2.

Introduction and Statement of the Problem

As the speeds of the machinery using journal bearings increased after the turn of the century, the interest in the development of journal bearing theory increased considerably. The users of such machinery were reporting large vibrational amplitudes under certain conditions of loading and speed which in turn caused large forces to be transmitted to the system foundation and the system's component parts.

Newkirk [1][1] reported in 1924 the first recorded instance of bearing instability. He demonstrated that under certain combinations of speed and loading, the journal center did not remain fixed as predicted by the steady-state Reynolds equation, but precessed or orbited about the equilibrium position at a speed approximately equal to half the rotational speed. This phenomenon was termed oil whip or whirl and is a self-excited motion. (See also references [2–10].)

A complete dynamical analysis of such a system requires that the hydrodynamic force terms be coupled to the dynamical equations of motion of the rotor (journal), including the external loading forces on the system and the unbalance of the journal. Figs. 1 and 2 show the typical journal bearing schematic, force balance, and unbalance representation which will be used in the following analysis.

The resulting equations of motion for the complete system are highly nonlinear and the stability characteristics have been examined primarily from a linearized or perturbation analysis about the equilibrium position of a balanced journal under unidirectional loading.

The bearing stability obtained from linearized theory only predicts the threshold of stability. It does not give any information as to the magnitude of the journal orbit when operation is above the whirl threshold speed. The linearized theory predicts that the journal motion will grow exponentially or become unbounded when the rotor is operated above the whirl threshold speed. In actuality, the journal motion is bounded and the motion forms limit cycles.

With the aid of the high-speed digital computer and the proper formulation of the hydrodynamic force expressions, the complete nonlinear motion of the journal bearing system may be obtained through the use of numerical methods for integrating the governing equations of motion.

In addition to the determination of the journal motion under arbitrary loading above and below the stability threshold, it is equally important that the bearing forces and the bearing dynamic transmissibility characteristics be determined. The results of such an analysis follows.

Analysis of the System

This section contains the derivation of the equations of motion for the journal bearing. Fig. 1 gives a schematic of a typical journal bearing. The clearance between the journal and bearing has been greatly exaggerated to clarify the representation of the bearing parameters. The journal center, o_j, is free to move about in the imaginary clearance circle depicted by the dashed circle in Fig. 1. The radial displacement of the journal center, o_j, from the bearing center, o_b, is denoted as the eccentricity, e, of the journal, and when divided by the clearance, c, the eccentricity ratio, ϵ, may then take

[1] Numbers in brackets designate References at end of paper.

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on values only from zero to unity. Unity represents bearing failure, while a value of zero has the journal perfectly centered in the bearing. See Fig. 2 for a typical force balance. It is therefore possible to represent the journal motion by a point moving about in a unit clearance circle, where all displacements are made dimensionless by dividing them by the clearance. This representation will be used extensively throughout the following analysis.

The Navier-Stokes equations can be expressed in vector notation as

$$\rho \frac{D\bar{u}}{Dt} = -\bar{\nabla} P + \bar{B} + \mu \cdot \left[\frac{1}{3}\bar{\nabla}(\bar{\nabla}\cdot\bar{u}) + \nabla^2\bar{u}\right] \tag{1}$$

For the purpose of this particular derivation, the incompressible fluid film between two flat plates of length l and width b, separated by some small distance $h = f(x, z)$, will be examined (see Fig. 3). If in addition the body forces are neglected, equation (1) may be expressed as follows:[2]

$$\rho \frac{D\bar{u}}{Dt} = -\bar{\nabla} P + \mu \dot{\nabla}^2\bar{u} \tag{2}$$

Furthermore, if the ratio of $h/1$ is restricted to be much less than unity, i.e., $h/1 \ll 1$, it may be concluded that the reduced Reynolds number, Re, is much much less than unity and it is hence possible to neglect the inertia force terms on the left of equation (2).

Imposing the conditions of

$$\rho = \text{constant}$$

Axial velocity of the bearing surfaces = 0.0

$$\partial h/\partial z = 0,$$

the following form of Reynolds' equation may be obtained from equation (2):

$$\frac{\partial}{\partial x}\left[\frac{h^3}{\mu}\frac{\partial P}{\partial x}\right] + \frac{\partial}{\partial z}\left[\frac{h^3}{\mu}\frac{\partial P}{\partial z}\right] = 12(v_2 - v_1) + 6(u_1 - u_2)\frac{\partial h}{\partial x} + 6h\frac{\partial}{\partial x}(u_1 + u_2) \tag{3}$$

[2] This equation can also be applied in journal bearing analysis for the case of a compressible fluid due to the order of magnitude of the term $\bar{\nabla}(\bar{\nabla}.\bar{u})$.

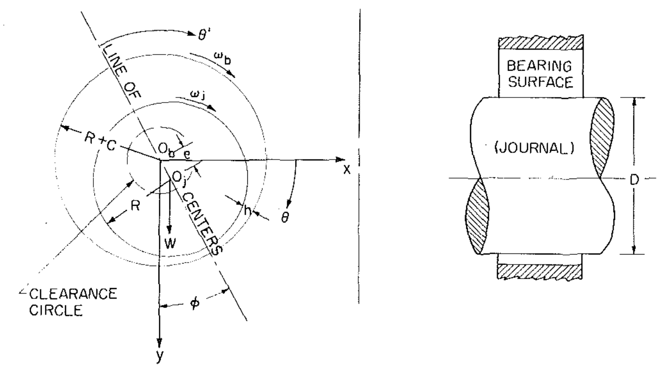

Fig. 1 Short journal bearing configuration

It is now desired to relate this solution to the geometry of the journal bearing. The first approach considers rotating coordinates, in which the film thickness may be expressed as (see Fig. 4)

$$h(\theta') = C(1 + \epsilon \cos\theta) \tag{4}$$

Linear combinations of the following components of motion will be considered.

(*a*) Rotation of journal about o_j, at ω_j.
(*b*) Rotation of bearing about o_b, at ω_b.
(*c*) Radial motion of o_j along the line of centers.
(*d*) Precession of o_j about o_b with angular velocity, $\omega_p = \dot{\phi}$.

If the film thickness is "unwrapped" the velocities due to (*a*) and (*b*) above may be expressed by the following components:

$$U_1 = (R + C)\omega_b \approx R\omega_b$$

$$U_2 = R\omega_j \cos\alpha \sim R\omega_j$$

$$V_1 = 0$$

$$V_2 = R\omega_j \sin\alpha \approx \alpha R\omega_j = \omega_j \frac{\partial h}{\sigma\theta'},$$

Since

$$\tan(\alpha) = \frac{dh}{dx} = \frac{1}{R}\frac{dh}{d\theta'}$$

and for $\alpha \ll 1$,

Nomenclature

$\bar{B}$ = body forces per unit volume
b = width of slider bearing
C_{ij} = damping coefficients
C, c = journal clearance
D = journal diameter
E_0, ES = eccentricity ratio calculated using the total resultant journal load for P in the Sommerfeld equation
E_μ, EMU = unbalance eccentricity ratio = e_μ/c
EN = ratio of rotating load angular speed = ω_{F0}/ω_j
e = radical journal center displacement, L
e_μ = unbalance eccentricity, L
ϕ_ϕ, ϕ_r = unit vectors referenced from the bearing center
ϕ_N, ϕ_t = unit vectors referenced from the instantaneous center of curvature
F_0 = magnitude of rotating load, F
F_x, F_y = force due to film in x and y coordinate directions respectively, F
$\bar{F}_x, \bar{F}_y$ = dimensionless forces from fluid film in x and y directions
g = acceleration of gravity, L/T^2
H = dimensionless film thickness = h/c
h = film thickness, L
h = stepping increment of independent variable
i, j, k = unit vectors in the fixed x-y-z coordinate directions
K_{ij} = stiffness coefficients, F/L
L = length of journal, L
l = length of slider, L
M_j, m_j = effective mass of journal, F-T^2/L
o_b = bearing center
o_j = journal center
P, p = pressure, F/L^2
P = projected load = $W/(L \times D)$, F/L^2
$\bar{P}$ = dimensionless pressure = $(p/((N_b + N_j) \times \mu)) \times (c/L)^2$
R, r = journal radius, L
Re = Reynolds number = UL/ν
Re* = modified Reynolds number = Re × $(h/1)^2$
S = Sommerfeld number = $\mu N/P(R/c)^2$, rev.
SS, S_S = short bearing Sommerfeld number (capacity number, Ocvirk number) = $S \times (L/D)^2$, rev.
t = time, T
U_i = velocity, L/T
$\bar{u}$ = vector representation of velocity, L/T
u = velocity component in x-coordinate direction, L/T
V_i = velocity, L/T
$\bar{V}_p$ = velocity of a point on the journal surface, L/T
$\bar{V}_Q$ = velocity of a point on the bearing surface, L/T
v = velocity component in the y-coordinate direction, L/T
W = effective weight of the journal, F
W_i = velocity, L/T
w = velocity component in z-coordinate direction, L/T
X = dimensionless displacement = x/c
$\dot{X}$ = dimensionless velocity = $\dot{x}/c\omega_j$
x, x_1 = displacement of journal in x-coordinate direction, L
x_2 = displacement of bearing in x-coordinate direction, L
x_i = solution at ith step of independent

$$\alpha \approx \frac{1}{R}\frac{dh}{d\theta'}$$

By neglecting the stretch effect in equation (3), the contributions due to rotation ω_b and ω_j are given by

$$\left[\frac{1}{6}\bar{\nabla}\cdot\left(\frac{h^3}{\mu}\bar{\nabla}P\right)\right]_{a,b} = (R\omega_b - R\omega_j)\frac{1}{R}\frac{\partial h}{\partial\theta'} + 2\omega_j\frac{\partial h}{\partial\theta'}$$

$$\left[\frac{1}{6}\bar{\nabla}\cdot\left(\frac{h^3}{\mu}\bar{\nabla}P\right)\right]_{a,b} = (\omega_b + \omega_j)\frac{\partial h}{\partial\theta'} \tag{5}$$

For the radial motion along the line of centers it can be shown that

$$U_2 = \dot{e}\sin\theta'$$
$$V_2 = \dot{e}\cos\theta'$$
$$U_1 = V_1 = 0$$

Therefore

$$\left[\frac{1}{6}\bar{\nabla}\cdot\left(\frac{h^3}{\mu}\bar{\nabla}P\right)\right]_c = (-\dot{e}\sin\theta')\frac{1}{R}\frac{\partial h}{\partial\theta'} + 2\dot{e}\cos\theta'$$

but by differentiating equation (4), it follows that

$$\left[\frac{1}{6}\bar{\nabla}\cdot\left(\frac{h^3}{\mu}\bar{\nabla}P\right)\right]_c = \dot{e}\left[\sin^2\theta'\cancelto{0}{\frac{e}{R}} + 2\cos\theta'\right]$$
$$= 2\frac{\partial h}{\partial t} \tag{6}$$

For precession, it is known that every point in the journal has velocity $e\dot{\phi}$ and is directed normal to the line of centers. Therefore the following velocity components are due to precession:

$$U_2 = -e\dot{\phi}\cos\theta'$$
$$V_2 = e\dot{\phi}\sin\theta'$$
$$U_1 = V_1 = 0$$

Substituting these expressions into equation (3) yields

$$\frac{1}{6}\left[\bar{\nabla}\cdot\left(\frac{h^3}{\mu}\bar{\nabla}P\right)\right]_d = \cancelto{0}{\frac{e}{R}}\dot{\phi}\cos\theta'\frac{\partial h}{\partial\theta'} + 2e\dot{\phi}\sin\theta'$$
$$= -2\dot{\phi}\frac{\partial h}{\partial\theta'} \tag{7}$$

Combining equations (5), (6), and (7) results in

$$\frac{1}{6}\left[\bar{\nabla}\cdot\left(\frac{h^3}{\mu}\bar{\nabla}P\right)\right] = (\omega_b + \omega_j - 2\dot{\phi})\frac{\partial h}{\partial\theta'} + 2\frac{\partial h}{\partial t} \tag{8a}$$

or

$$\frac{1}{6}\left[\frac{1}{R^2}\frac{\partial}{\partial\theta'}\left(\frac{h^3}{\mu}\frac{\partial P}{\partial\theta'}\right) + \frac{\partial}{\partial z}\left(\frac{h^3}{\mu}\frac{\partial P}{\partial z}\right)\right] = (\omega_b + \omega_j - 2\dot{\phi})\frac{\partial h}{\partial\theta'} + 2\frac{\partial h}{\partial t} \tag{8b}$$

This expression, i.e., equation (8), is the Reynolds equation for a plain journal bearing using rotating coordinates, where θ is the angle measured from the line of centers in the positive coordinate direction. This form of Reynolds' equation is the expression that

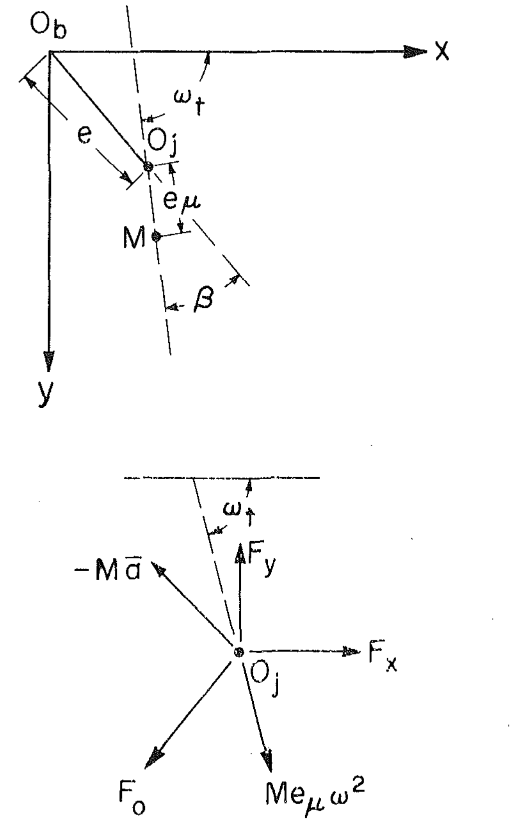

Fig. 2 Unbalance representation (above) and a typical force balance

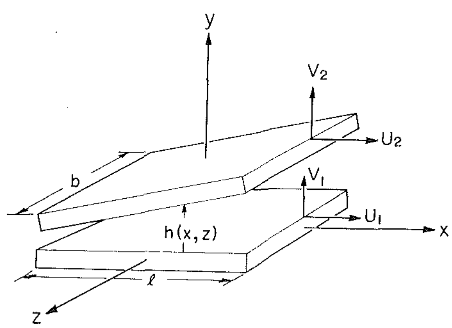

Fig. 3 The plane slider bearing

Nomenclature

variable

Y = dimensionless displacement = y/c

$\dot{Y}$ = dimensionless velocity = $\dot{y}/c\omega_j$

y, y_1 = displacement of journal in y-coordinate direction, L

y_2 = displacement of bearing in the y-coordinate direction, L

$\bar{Z}$ = dimensionless distance along axial or z-direction = z/L

z = distance along z-coordinate, L

α = ratio of angular velocities = $\omega_j/(\omega_b + \omega_j)$

α = angle between velocities $R\omega_j$ and $(R + C)\omega_b$, $\sim(1/R)(\partial h/\partial\theta)$

β = phase angle between the journal displacement vector and the unbalance eccentricity vector, deg

ϵ, = eccentricity ratio = e/c

ϵ_0 = eccentricity calculated from equation of S_S

η = ratio of rotating load velocities = ω_{F0}/ω_j

θ' = angular distance from positive line of centers in rotating coordinate set

θ = angular distance from the positive x-axis in the fixed x-y coordinate set

$\dot{\theta}$ = instantaneous angular velocity about the center of curvature, T^{-1}

λ = root to the characteristic equation of journal equations of motion, T^{-1}

μ = viscosity, $F\text{-}T/L^2$

ν = kinematic viscosity, = μ/ρ, L_2/T

ρ = density, F/gL^3

ρ = instantaneous radius of curvature, L

$\dot{\phi}$ = whirl velocity about bearing center, T^{-1}

ϕ = attitude angle, deg

Ω, ω_j = journal angular velocity, T^{-1}

Ω_S = speed parameters, = $\omega_j/\sqrt{W_T/m_jc}$

ω_b = bearing angular velocity, T^{-1}

ω_S = speed parameter, = $\omega_j/\sqrt{g/c}$, T^{-1}

ω = angular speed defined as $\omega_b + \omega_j$, T^{-1}

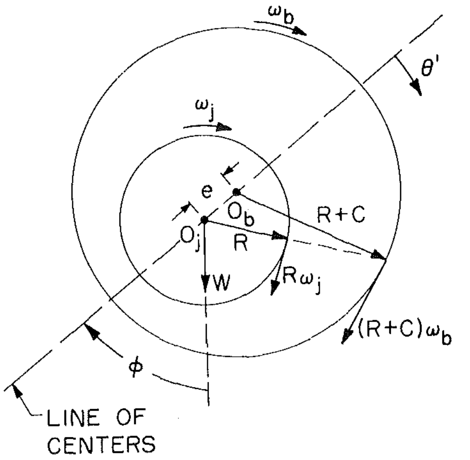

Fig. 4(a). Journal bearing in rotating coordinates

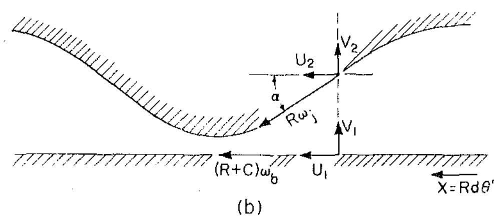

Fig. 4(b). Journal bearing profile unwrapped

most of the work in this field is based on. To avoid coordinate transformations, the following derivation in fixed Cartesian coordinates is presented.

The following unit vectors will be used to express the derived velocity components (see Fig. 5):

$$\begin{aligned} \mathbf{i} &= -\cos\theta\,\eta_R - \sin\theta\,\eta_\theta \\ \mathbf{j} &= -\sin\theta\,\eta_R + \cos\theta\,\eta_\theta \\ \eta_R &= -\cos\theta\mathbf{i} - \sin\theta\mathbf{j} \\ \eta_\theta &= -\sin\theta\mathbf{i} + \cos\theta\mathbf{j} \end{aligned} \tag{9}$$

The velocities of the bearing and journal centers are as follows:

$$V_{0b} = \dot{x}_1\mathbf{i} + \dot{y}_1\mathbf{j}$$
$$V_{0j} = \dot{x}_2\mathbf{i} + \dot{y}_2\mathbf{j}$$

The velocity of point "Q" is:

$$\mathbf{V}_Q = \mathbf{V}_{0b} + \omega_b \times \mathbf{R} = \dot{x}_1\mathbf{i} + \dot{y}_1\mathbf{j} + R\omega_b\,\eta_\theta$$

and also

$$\mathbf{V}_Q = V_1\,\eta_R + u_1\,\eta_\theta$$

or

$$V_1 = \mathbf{V}_Q\cdot\eta_R = -\dot{x}_1\cos\theta - \dot{y}_1\sin\theta \tag{10}$$

and

$$u_1 = \mathbf{V}_Q\cdot\eta_\theta = \omega_b R - \dot{x}_1\sin\theta + \dot{y}_1\cos\theta \tag{11}$$

For point "P," it is necessary to relate the velocities to V_2 and U_2.

The $R\omega_j$ component is not in line with the $R\omega_b$ component; they differ by the angle α. For small displacements these are related small angles and it is thus possible to approximate α as follows:

$$\tan\alpha \approx \alpha = \frac{\Delta h}{\Delta x} \sim \frac{\partial h}{\partial x} = \frac{1}{R}\frac{\partial h}{\partial\theta}$$

Also

$$\tan\alpha \approx \sin\alpha \approx \alpha$$

For the theta direction

$$Vel_\theta = R\omega_j\cos\alpha \approx R\omega_j$$

and for the radial direction

$$Vel_r = R\omega_j\sin\alpha \approx \omega_j\frac{\partial h}{\partial\theta}$$

So, it is now possible to express the velocity of point "P" as

$$\mathbf{V}_P = \dot{x}_2\mathbf{i} + \dot{y}_2\mathbf{j} + R\omega_j\,\eta_\theta + \omega_j\frac{\partial h}{\partial\theta}\,\eta_R$$

Therefore

$$u_2 = \mathbf{V}_P\cdot\eta_\theta = -\dot{x}_2\sin\theta + \dot{y}_2\cos\theta + R\omega_j \tag{12}$$

and

$$V_2 = \mathbf{V}_P\cdot\eta_R = -\dot{x}_2\cos\theta - \dot{y}_2\sin\theta + \omega_j\frac{\partial h}{\partial\theta} \tag{13}$$

By neglecting the stretch effect of equation (3), i.e., $h \times (\partial/\partial x)(U_1 + U_2)$, substitution of the appropriate velocities into that equation gives

$$\begin{aligned} \frac{1}{6}[\bar{\nabla}\cdot(\frac{h^3}{\mu}\bar{\nabla}P)] &= \frac{1}{R}(u_1 - u_2)\frac{\partial h}{\partial\theta} + 2(V_2 - V_1) \\ &= (\omega_b + \omega_j)\frac{\partial h}{\partial\theta} + (\dot{x}_2 - \dot{x}_1)(-2\cos\theta + \frac{1}{R}\frac{\partial h}{\partial\theta}\sin\theta) \\ &\quad - (\dot{y}_2 - \dot{y}_1)(2\sin\theta + \frac{1}{R}\frac{\partial h}{\partial\theta}\cos\theta) \end{aligned}$$

In addition, by neglecting the $\frac{1}{R}\frac{\partial h}{\partial\theta}$ terms,

$$\frac{1}{6}[\bar{\nabla}\cdot(\frac{h^3}{\mu}\bar{\nabla}P)] = (\omega_b + \omega_j)\frac{\partial h}{\partial\theta} + (\dot{x}_2 - \dot{x}_1)(-2\cos\theta) - (\dot{y}_2 - \dot{y}_1)(2\sin\theta) \tag{14}$$

But for small deflection, the film thickness, h, is given as (see Fig. 5)

$$h = c - (x_2 - x_1)\cos\theta - (y_2 - y_1)\sin\theta \tag{15}$$

Since from equation (4) we can write

$$\begin{aligned} h(\theta) &= c - e\cos(\theta - (90 - \phi)) \\ &= c - e\cos\theta\sin\phi - e\sin\theta\cos\phi \end{aligned}$$

where

$$e\sin\phi = x_2 - x_1,\quad e\cos\phi = y_2 - y_1$$

It is now possible to rewrite equation (14) as follows:

$$\begin{aligned} \frac{1}{6}[\frac{1}{R^2}\frac{\partial}{\partial\theta}(\frac{h^3}{\mu}\frac{\partial P}{\partial\theta}) + \frac{\partial}{\partial z}(\frac{h^3}{\mu}\frac{\partial P}{\partial z})] &= (\omega_b + \omega_j)\frac{\partial h}{\partial\theta} + 2\frac{\partial h}{\partial t} \\ &= (\omega_b + \omega_j)[(x_2 - x_1)\sin\theta - (y_2 - y_1)\cos\theta] \\ &\quad - (\dot{x}_2 - \dot{x}_1)(2\cos\theta) - (\dot{y}_2 - \dot{y}_1)(2\sin\theta) \end{aligned} \tag{16}$$

It must be remembered that in this equation the "θ" is measured from the fixed x-axis and should not be confused with the rotating coordinate set where the θ' is measured from the line of centers.

Two basic approaches to the solution of equation (16) have been reported in the literature. If it is assumed that the journal bearing is very long, then it is possible to neglect the fluid flow and pressure gradients along the z-axis and hence reduce equation (16) to

$$\frac{1}{6R^2}\frac{\partial}{\partial\theta}(\frac{h^3}{\mu}\frac{\partial P}{\partial\theta}) = (\omega_b + \omega_j)\frac{\partial h}{\partial\theta} + 2\frac{\partial h}{\partial t} \tag{17}$$

This solution is known as the long bearing solution and was first solved by Sommerfeld, who used an adroite substitution and succeeded in integrating equation (11).

On the other hand, if it is assumed that the bearing is relatively short, the appropriate approach is to neglect the flow in the theta direction due to pressure gradients and arrive at

$$\frac{1}{6}\frac{\partial}{\partial z}(\frac{h^3}{\mu}\frac{\partial P}{\partial z}) = (\omega_b + \omega_j)\frac{\partial h}{\partial\theta} + 2\frac{\partial h}{\partial t} \tag{18}$$

which is known as the governing equation for the short bearing so-

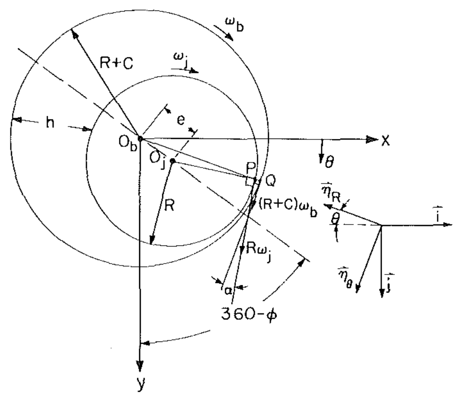

Fig. 5 Journal bearing in fixed Cartesian coordinates

lution. This approach to Reynolds' equation is the basis for the computer program and resulting analysis to be presented in the following sections.

To have a better understanding of when the given assumption is a valid one, the solid curves in Fig. 6 were drawn from data for a finite full journal bearing (reference [11], pp. 86–88). Those data were reported to have come from digital computer solutions of the general Reynolds equation. In addition, the corresponding Sommerfeld number obtained from the short bearing solution is plotted for the same length to diameter ratios. It is easy to see that the assumption is very good for L/D ratios of ½ or less, or for $L/R \leq 1$. It is also apparent that more deviation exists at larger eccentricity values for $L/R \geq 1$, whereas for smaller values the agreement is very good indeed.

The reason for the deviation in the short bearing solution has been explained by Ocvirk [4] to arise from the higher pressures predicted due to neglecting the theta pressure flow in the journal. However, by realizing the limitations of the solution there should be no confusion about the results and conclusions obtained from the given theory.

Dynamical Equations of Motion

The Reynolds equation has been derived in the previous section for the plane slider and by proper substitution and assumptions, it has been reduced to the following equation which is valid for a "short" journal bearing:

$$\frac{\partial}{\partial z}\left(\frac{h^3}{6\mu}\frac{\partial P}{\partial z}\right) = (\omega_b + \omega_j)\frac{\partial h}{\partial \theta} + 2\frac{\partial h}{\partial t} \tag{19}$$

In fixed coordinates, the film thickness, h, is given by

$$h = c - x\cos\theta - y\sin\theta \tag{20}$$

This equation is valid for a journal bearing that has no axial misalignment and was derived by considering small motions in the x and y directions to be linearly related. In addition, by limiting the motion of the bearing to rotation, ω_b, all displacements will be relative to the bearing center, o_b.

Equation (19) can be integrated directly and by applying the boundary conditions

$$P(\theta, 0) = P(\theta, L) = 0 \tag{21}$$

to evaluate the two constants of integration; the following equation results:

$$P(\theta, z) = \frac{3\mu z(z-L)}{h^3}\left[(\omega_b + \omega_j)\frac{\partial h}{\partial \theta} + 2\frac{\partial h}{\partial t}\right] \tag{22}$$

From (20)

$$\frac{\partial h}{\partial \theta} = x\sin\theta - y\cos\theta \tag{23}$$

and

$$\frac{\partial h}{\partial t} = -\dot{x}\cos\theta - \dot{y}\sin\theta \tag{24}$$

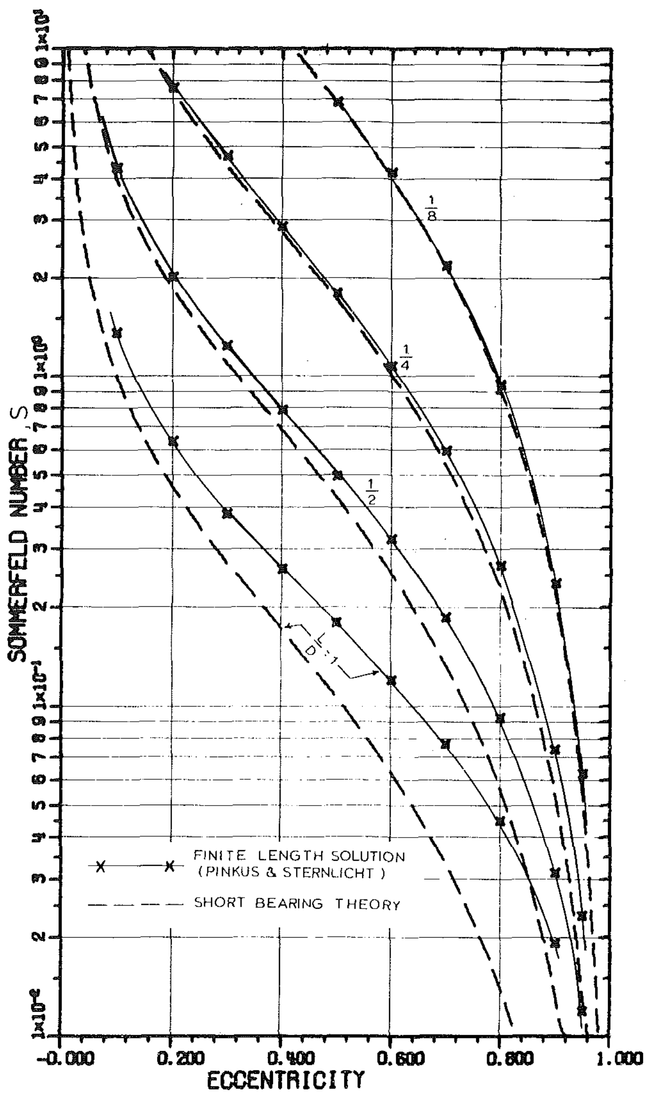

Fig. 6 Comparison of finite length and short bearing solutions

The increment of force on the journal is given as

$$\Delta\mathbf{F} = P(\theta, z)Rd\theta dz\ \eta_R$$

where

$$\eta_R = -\cos\theta\mathbf{i} - \sin\theta\mathbf{j}$$

Therefore

$$\Delta\mathbf{F}_x = (\Delta\mathbf{F}\cdot\mathbf{i})\mathbf{i} = -[P(\theta, z)Rd\theta dz\cos\theta]\mathbf{i} \tag{25}$$

and

$$\Delta\mathbf{F}_y = (\Delta\mathbf{F}\cdot\mathbf{j})\mathbf{j} = -[P(\theta, z)Rd\theta dz\sin\theta]\mathbf{j} \tag{26}$$

where the total force component is

$$F_x = -\iint P(\theta, z)R\cos\theta d\theta dz$$

and

$$F_y = -\iint P(\theta, z)R\sin\theta d\theta dz$$

The result of integrating over the length of the bearing and substituting equations (23) and (24) leads to the following equations:

$$\begin{Bmatrix}F_x\\F_y\end{Bmatrix} = \frac{\mu R L^3}{2}\cdot \times \int_0^{2\pi}\frac{(\omega_j+\omega_b)(x\sin\theta - y\cos\theta) - 2(\dot{x}\cos\theta + \dot{y}\sin\theta)}{(c - x\cos\theta - y\sin\theta)^3} \times \begin{Bmatrix}\cos\theta\\\sin\theta\end{Bmatrix}d\theta \tag{27}$$

The integral of equation (27) must be integrated very carefully since a subambient pressure will not be permitted to exist in the fluid film. This follows from reports on experimental test rigs as discussed by reference [11], p. 435.

It is possible to put equation (27) into dimensionless form if the following representation is used:

$$X = \frac{x}{c},\ Y = \frac{y}{c},\ \omega = \omega_b + \omega_j,\ \dot{X} = \frac{\dot{x}}{c\omega_j},\ \dot{Y} = \frac{\dot{y}}{c\omega_j},\ \alpha = \frac{\omega_j}{\omega}$$

then (27) becomes

$$\left\{\begin{matrix}F_x\\F_y\end{matrix}\right\} = \frac{\mu L^3 R\omega}{2c^2}\int_0^{2\pi}\frac{(X - 2\dot{Y}\alpha)\sin\theta - (Y + 2\dot{X}\alpha)\cos\theta}{(1 - X\cos\theta - Y\sin\theta)^3} \times \left\{\begin{matrix}\cos\theta\\ \sin\theta\end{matrix}\right\}d\theta \quad (28)$$

The equations of motion of the journal can be derived by considering Newton's second law:

$$(\Sigma F)_{x,y} = m_j a_{j_{x,y}} \quad (29)$$

A position vector to the center of mass of the journal is

$$P_m = (x + e_\mu \cos\omega_j t)\mathrm{i} + (y + e_\mu \sin\omega_j t)\mathrm{j} \quad (30)$$

where the mass center is located a radial distance of e_μ from the geometric center of the journal. Then the acceleration is given by

$$a_m = (\ddot{x} - e_\mu\omega_j^2\cos\omega_j t)\mathrm{i} + (\ddot{y} - e_\mu\omega_j^2\sin\omega_j t)\mathrm{j} \quad (31)$$

therefore

$$m_j(\ddot{x} - e_\mu\omega_j^2\cos\omega_j t) = (\Sigma F)x \quad (32a)$$

$$m_j(\ddot{y} - e_\mu\omega_j^2\sin\omega_j t) = (\Sigma F)y \quad (32b)$$

where the right-hand side of the given equations represents all loading on the journal, including the forces given by equation (27) that are developed in the fluid film. By letting

$$\omega_j = \Omega,\ \Omega t = T,\ \frac{e_\mu}{c} = E_\mu$$

and dividing through by $m_j c\Omega^2$, the equations become

$$\frac{d^2X}{dT^2} = E_\mu\cos T + \frac{F_0}{m_j c\Omega^2}\cos(\eta T) + \frac{\mu R L^3\omega}{2m_j c^3\Omega^2}\bar{F}_x(X, Y, \dot{X}, \dot{Y}) + \frac{FX}{m_j c\Omega^2} + \ldots \quad (33)$$

$$\frac{d^2Y}{dT^2} = E_\mu\sin T + \frac{F_0}{m_j c\Omega^2}\sin(\eta T) + \frac{\mu R L^3\omega}{2m_j c^3\Omega^2}\bar{F}_y(X, Y, \dot{X}, \dot{Y}) + \frac{FY}{m_j c\Omega^2} + \ldots \quad (34)$$

where:

F_0 = rotating load at some multiple of the journal frequency.

$FX = FY$ = constant loading in x and y direction, respectively.

Also, other loading can be added as noted by + . . .

The given equations are for a vertical journal bearing since we have not included the gravity loading in the equations of motion.

Journal Precession Rate and Radius of Curvature

When a journal bearing is acted upon by some unbalance force or other cyclic forcing function the journal tends to move in an orbit due to the forces acting upon it. If the orbit encloses the center of the journal then it possible to think of the distance from the bearing center to the journal center as the radius of the path and the angular velocity with respect to the bearing center as the whirl frequency. Referring to Fig. 7, these quantities would be radius e and angular velocity $\dot{\phi}$.

These values may be expressed in terms of the displacements and velocities, $x, y, \dot{x}, \dot{y}$ by the following procedure. The velocity of the journal center may be expressed as

$$\mathbf{V}_j = \dot{x}\mathrm{i} + \dot{y}\mathrm{j} = \dot{e}\,\mathrm{e}_r + e\dot{\phi}\,\mathrm{e}_\phi$$

The relation among the unit vectors is as follows:

$$\mathrm{e}_r = \cos\phi\mathrm{i} + \sin\phi\mathrm{j}$$

$$\mathrm{e}_\phi = -\sin\phi\mathrm{i} + \cos\phi\mathrm{j}$$

$$x = e\cos\phi,\ y = e\sin\phi$$

So

$$\mathbf{V}_j \cdot \mathrm{e}_\phi = e\dot{\phi} = \dot{x}\mathrm{i}\cdot\mathrm{e}_\phi + \dot{y}\mathrm{j}\cdot\mathrm{e}_\phi = -\sin\phi\dot{x} + \cos\phi\dot{y}$$

Therefore

$$\dot{\phi} = \frac{e(\cos\phi\dot{y} - \sin\phi\dot{x})}{e\ e} = \frac{x\dot{y} - y\dot{x}}{e^2}$$

or since $e^2 = x^2 + y^2$ then,

$$\dot{\phi} = \frac{x\dot{y} - y\dot{x}}{x^2 + y^2}$$

In dimensionless form

$$\frac{\dot{\phi}}{\Omega} = \frac{X\dot{Y} - Y\dot{X}}{X^2 + Y^2} \quad (35)$$

where

$$X = x/c,\ Y = y/c$$

and

$$\dot{X} = \dot{x}/c,\ \dot{Y} = \dot{y}/c,\ \text{with } \Omega = \omega_j \text{ as before.}$$

The radius, e, is given by

$$e = c \times \sqrt{X^2 + Y^2} \quad (36)$$

However, it is obvious that (36) has little meaning if the journal orbit does not enclose the bearing center, o_b.

The equations for the instantaneous radius of curvature ρ and angular velocity $\dot{\theta}$ will now be developed.

The velocity can be expressed for this purpose as

$$\mathbf{V}_j = V\mathrm{e}_t = \rho\dot{\theta}\mathrm{e}_t$$

where

ρ = instantaneous radius of curvature

$\dot{\theta}$ = instantaneous angular velocity about o_b

The acceleration is expressed as

$$\mathrm{a}_j = \frac{d}{dt}(V\mathrm{e}_t) = \dot{V}\mathrm{e}_t + V\frac{d\mathrm{e}_t}{dt} = a_t\mathrm{e}_t + \frac{V^2}{\rho}\mathrm{e}_N$$

But since

$$\mathrm{a}_j = \ddot{x}\mathrm{i} + \ddot{y}\mathrm{j}$$

and

$$\mathrm{a}_j\cdot\mathrm{e}_N = \frac{V^2}{\rho} = \ddot{x}\mathrm{i}\cdot\mathrm{e}_N = \ddot{y}\mathrm{j}\cdot\mathrm{e}_N \quad (37)$$

The new unit vectors are determined as follows:

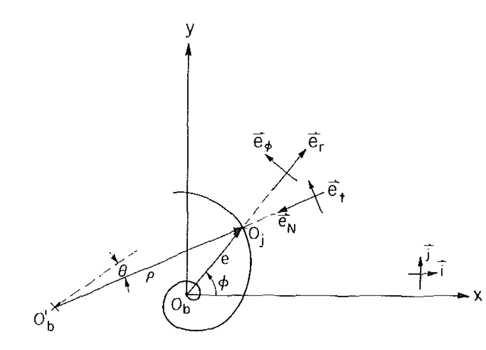

Fig. 7 Typical journal trajectory illustrating the instantaneous radius of curvature

$$\mathbf{V} = V\boldsymbol{\phi}_t = \dot{x}\mathbf{i} + \dot{y}\mathbf{j}$$

therefore

$$\boldsymbol{\phi}_t = \frac{1}{V}(\dot{x}\mathbf{i} + \dot{y}\mathbf{j})$$

and

$$\boldsymbol{\phi}_n = \mathbf{k}x\boldsymbol{\phi}_t = \frac{1}{V}(\dot{x}\mathbf{j} - \dot{y}\mathbf{i})$$

Hence

$$\mathbf{i} \cdot \boldsymbol{\phi}_n = -\dot{y}/V$$

and

$$\mathbf{j} \cdot \boldsymbol{\phi}_n = \dot{x}/V$$

So, from equation (38),

$$V^2/\rho = \frac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{V}$$

Solving for ρ and $\dot{\theta} = V/\rho$ results in

$$\rho = V^3/(\ddot{y}\dot{x} - \ddot{x}\dot{y}) \qquad (38)$$

and

$$\dot{\theta} = \frac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{\dot{x}^2 + \dot{y}^2} \qquad (39)$$

since

$$V^2 = \dot{x}^2 + \dot{y}^2$$

In dimensionless form, the instantaneous whirl ratio is given by

$$\frac{\dot{\theta}}{\Omega} = \frac{\ddot{Y}\dot{X} - \ddot{X}\dot{Y}}{\dot{X}^2 + \dot{Y}^2} \qquad (40)$$

This expression is meaningful for any orbit path the journal might traverse and will be easily calculated since it involves quantities readily available in the method of solution of the journal orbit.

The Fluid Film Pressure Profile

The pressure in the fluid film was given in fixed coordinates by equation (22). Substituting equations (23), (24) into equation (22) and expressing the result in dimensionless form gives

$$\bar{P} = \frac{P(\theta, z)}{\mu(N_b' + N_j')}\left(\frac{C}{L}\right)^2 = 6\pi\bar{Z}(1 - \bar{Z})$$
$$\times \left[\frac{-X\sin\theta + Y\cos\theta + 2\alpha(\dot{X}\cos\theta + \dot{Y}\sin\theta)}{(1 - X\cos\theta - Y\sin\theta)^3}\right] \qquad (41)$$

where

$$\bar{Z} = \frac{z}{L}, \quad \alpha = \frac{\omega_j}{\omega_b + \omega_j}, \quad X = \frac{x}{c}, \quad \dot{X} = \frac{1}{c\omega_j}\frac{dx}{dt}$$

If no values of negative pressure are allowed to exist in the fluid film, then all P's less than zero are equated to zero. The given equation was programmed on the digital and the results plotted via an automatic plotter. Various cases were considered and are presented as Figs. 8–12.

The values of the dimensionless displacements and velocities are given at the top of the figures. An end view of the section at the bearing midspan is given in the upper left corner with the pressure profile represented as radial lines. The center figure is a three-dimensional plot of the "unwrapped" pressure profile. At the bottom of the figure is the film thickness, H, plotted versus angular distance, θ.

The maximum dimensionless pressure increases as the journal moves from near the center (Fig. 8) out to $X = 0.2$, $Y = -0.10$ (Fig. 9). Fig. 10 shows the uncavitated pressure surface for the case of Fig. 9. The negative pressures cannot be sustained in the fluid film and therefore cavitates. In Fig. 11 the case of $X = 0.5$, $Y = -0.7$ is given in small velocity which results in a slight increase of peak pressure above that for the static value ($\bar{P}_{\max} = 210.38$).

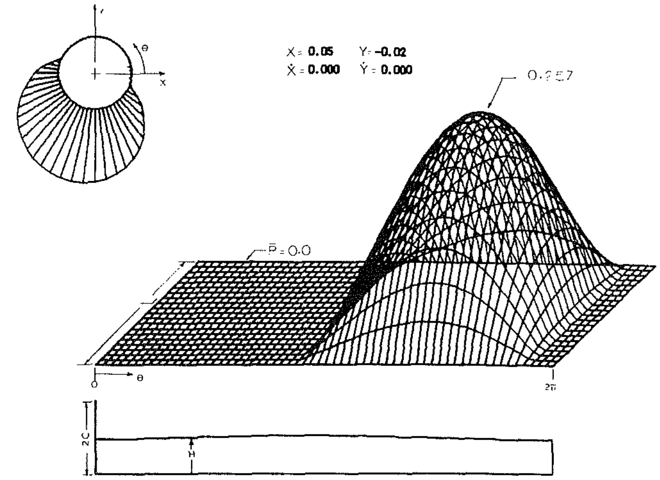

Fig. 8 Pressure profile, pressure surface, and film thickness for $X = 0.05$, $Y = -0.02$, $\dot{X} = \dot{Y} = 0$

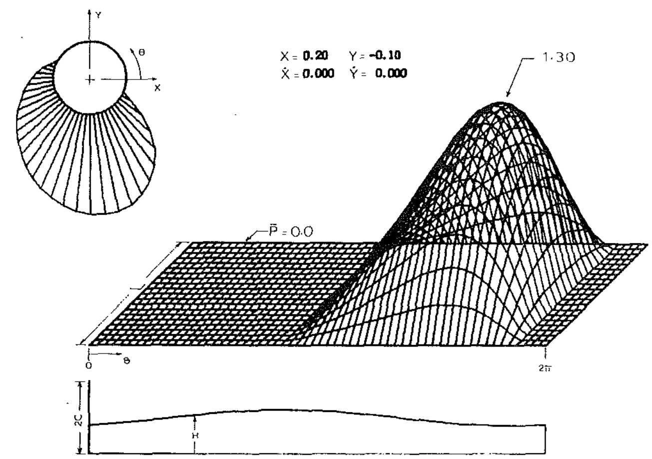

Fig. 9 Pressure profile, pressure surface, and film thickness for $X = 0.2$, $Y = -0.10$, $\dot{X} = \dot{Y} = 0$

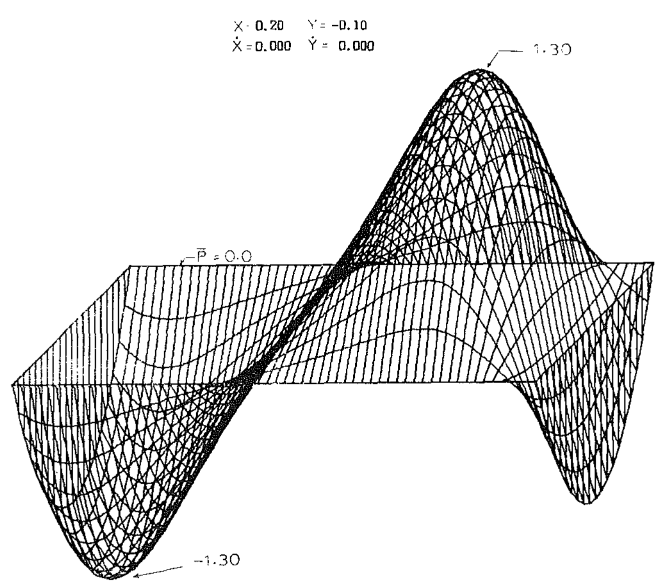

Fig. 10 Uncavitated pressure profile for $X = 0.2$, $Y = -0.10$, $\dot{X} = \dot{Y} = 0$

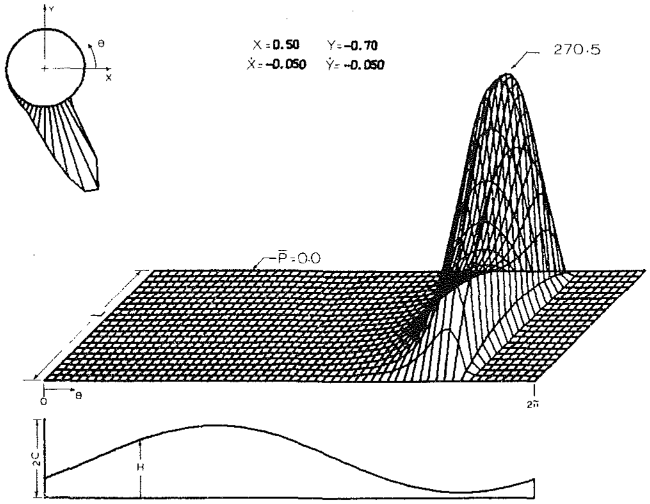

Fig. 11 Pressure profile, pressure surface, and film thickness for $X = 0.5$, $Y = -0.7$, $\dot{X} = \dot{Y} = -0.05$

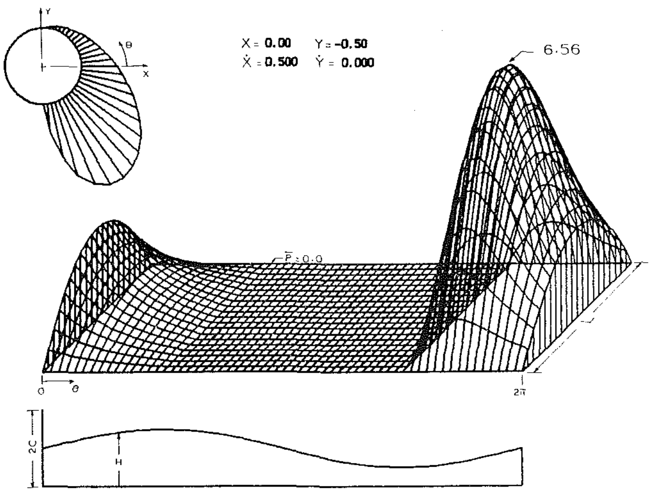

Fig. 12 Pressure profile, pressure surface, and film thickness for $X = \dot{Y} = 0.0$, $\dot{X} = -Y = 0.5$

Fig. 12 has been given the condition of synchronous whirl[3] at an eccentricity of 0.5. All cases plotted were for $\alpha = 1$, i.e., $\omega_b = 0$.

If it is assumed that gravity is acting in the negative y-coordinate direction, then the term $(-g/c\Omega 2)$ must be added to (34) to account for this effect.

The solution of equations (33) and (34) will give the journal orbit as a function of dimensionless time, T. Numerical methods will be used to integrate these equations of motion forward in time.

Stability Analysis for Small Displacements

The equation for the fluid-film force components has been derived for the short bearing model and is given by equation (28). It is now possible to obtain from this equation the stiffness and damping coefficients which are given as

$$K_{ij} = -\frac{\partial F_i}{\partial x_j};\ i = 1, 2;\ j = 1, 2 \tag{42}$$

$$C_{ij} = -\frac{\partial F_i}{\partial \dot{x}_j};\ i = 1, 2;\ j = 1, 2 \tag{43}$$

These coefficients can now be inserted into the equation of motion of the journal and a stability analysis performed. Plots of the resulting dimensionless values used in the analysis are presented in Figs. 13(a) and 13(b). The equation to be examined is given as follows:

$$\ddot{X}_i + \bar{C}_{ij}\dot{X}_j + \bar{K}_{ij}X_j = 0$$

where

$$\bar{M}_J = \frac{mc\omega_i^2}{W},\ \bar{K}_{ij} = \frac{CK_{ij}}{\bar{M}_J W},\ \bar{C}_{ij} = \frac{\omega CC_{ij}}{\bar{M}_J W} \tag{44}$$

The assumed solution is of the form

$$X_1 = Ae^{\lambda t},\ X_2 = Be^{\lambda t}$$

Making these substitutions results in the following equations:

$$\begin{bmatrix} \lambda^2 + \bar{C}_{11}\lambda + \bar{K}_{11} & \bar{C}_{12}\lambda + \bar{K}_{12} \\ \bar{C}_{21}\lambda + \bar{K}_{21} & \lambda^2 + \bar{C}_{22}\lambda + \bar{K}_{22} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{45}$$

By expanding the determinant of coefficients the following fourth order equation is obtained:

$$\lambda^4 + (\bar{C}_{11} + \bar{C}_{22})\lambda^3 + (\bar{K}_{11} + \bar{K}_{22} + \bar{C}_{11}C_{22} - \bar{C}_{12}\bar{C}_{21})\lambda^2 + (\bar{K}_{11}\bar{C}_{22} + \bar{K}_{22}C_{11} - \bar{K}_{12}\bar{C}_{21} - \bar{K}_{21}\bar{C}_{12})\lambda + (\bar{K}_{22}\bar{K}_{11} - \bar{K}_{12}\bar{K}_{21}) = 0 \tag{46}$$

In general terms the characteristic equation can be expressed as

$$\sum_{I=0}^{N} A_{N-I}\lambda^I = 0 \tag{47}$$

For $N = 4$

$$A_4 + A_3\lambda + A_2\lambda^2 + A_1\lambda^3 + A_0\lambda^4 = 0$$

For $A_i > 0$, the stability criteria may be expressed

$$A_1A_2A_3 > A_4A_1^2 + A_0A_3^2 \tag{48}$$

A stability analysis has been performed by the approach just described in reference [12]. That analysis considered stability about the equilibrium eccentricity and attitude angle considering the journal center to be initially at rest. The stability map resulting from that study is shown in Fig. 14 and is comparable to that of Badgley and Booker [13] who examined orbital plots for the journal center to determine whether or not the system was stable (see Fig. 15).

The criterion for instability that they used was an increasing radius arm as the orbit tracked out the journal center path.

These approaches to the problem of stability have only considered the horizontal, unloaded journal. A loaded journal will be shown to exhibit a greater area of stability on the stability map, while an unloaded vertical journal will be unstable over the entire range of the map. These are important facts that are not obvious from a plot such as Fig. 14, or the similar plots of Badgley and Booker, Fig. 15.

The analysis presented by Reddi [14] for the 180 deg long bearing, with end leakage considered, has given a lower threshold speed than the stability analysis using the short bearing equations. The threshold curve resulting from their analysis has been converted into the parameter used by Badgley and Booker for the special case that the loading is due to the weight of the journal (rotor) only. The Reddi-Trumpler threshold speed is less than that of Badgley-Booker but the limit of eccentricity at which the journal is completely stable is very nearly the same. This value is in agreement with Hori who gave 0.8 as the upper limit of eccentricity past which the journal is always stable.

In all of the various analyses of stability, only the threshold speed of unstable motion is predicted. In an actual bearing operated above the stability threshold speed, the journal does not fail but forms a finite limit cycle which increases with speed.

Summary

The equations necessary to calculate the transient response of a rotor supported in fluid-film bearings have been presented. A fixed

[3] The journal center is precessing in the bearing at the journal rotational angular velocity.

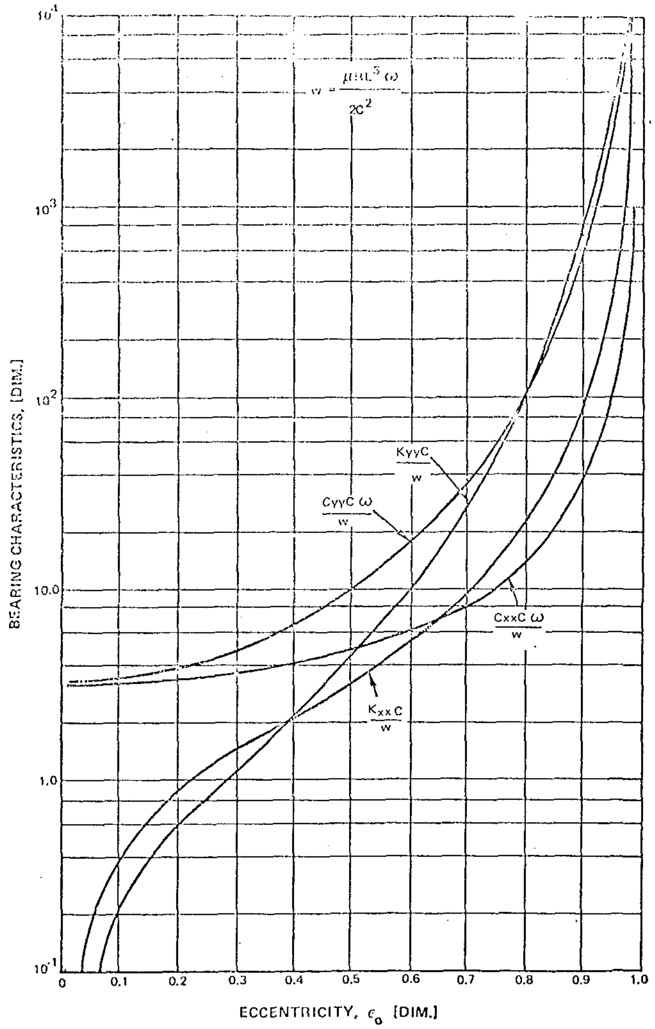

Fig. 13*(a)* Dimensionless direct stiffness and damping coefficients for the short journal bearing

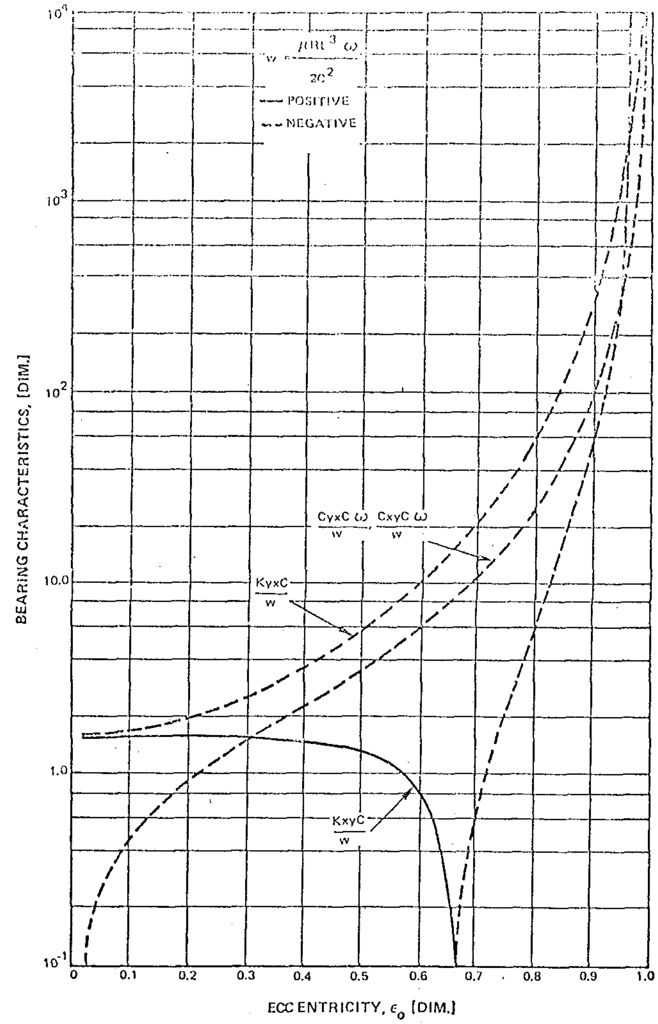

Fig. 13*(b)* Cross-coupled stiffness and damping coefficients for the short journal bearing

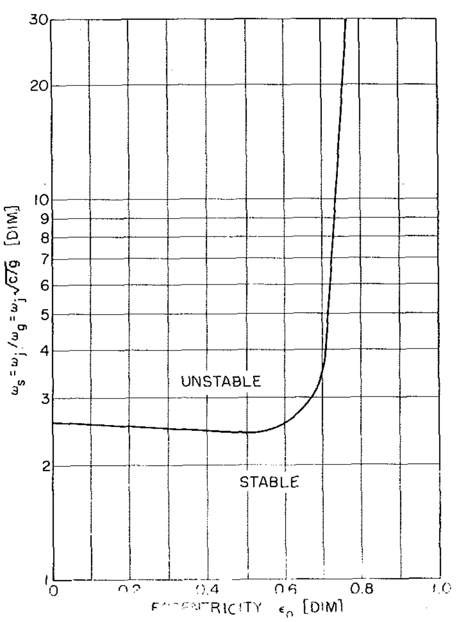

Fig. 14 Stability for the unloaded short journal bearing

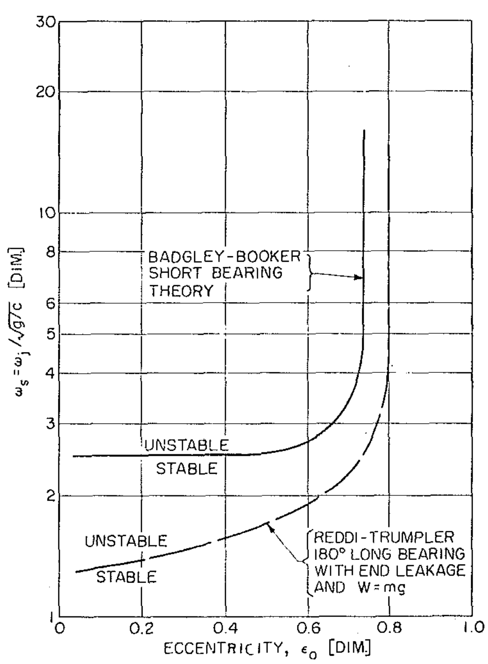

Fig. 15 Stability for the unloaded journal bearing (comparison of Reddi-Trumpler and Badgley-Booker stability maps)

Cartesian reference frame allows the bearing response to be calculated directly using the fixed reference coordinate parameters of the rotor shaft.

The results of a stability analysis using the fluid-film bearing characteristics obtained from the theory presented agrees with other published results in the literature. The results of an extensive study of journal bearing response to imbalance both below and above the stability threshold speed will be presented in Part 2. Examples of response to external nonsynchronous forcing functions will also be presented in Part 2.

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