A Simple Way to Measure Mass Moments of Inertia

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A simple rotational pendulum method to measure the radii of gyration or mass moments of inertia of a rotor and other assemblies is described.

In mechanical dynamic problems, it is often necessary to know the radii of gyration or equivalent mass moments of inertia for components and assemblies. Using the rotational pendulum technique described here, one can easily measure the radii of gyration about the polar and diametric axes of any rigid rotor without requiring a special fixture. The principles employed are also applicable to more complicated assemblies such as aircraft, boats and cars, where the radius of gyration and vehicle maneuverability are of interest. This description focuses on rotors.

The relative values of polar and diametric radii of gyration characterize some dynamic behavior and stability of spinning rotors. When the ratio of polar to diametric radii of gyration approaches unity, the spinning rotor may exhibit undesirable dynamic behavior. Consequently, prior to high-speed spin testing the rotor or otherwise operating the assembly, it is desirable to have a simple and inexpensive procedure which directly measures the radii of gyration of existing hardware. These data permit the technician to estimate the rotor dynamic behavior or to identify potential problems before committing to operation.

If sufficient part information is available, such as: dimensions, geometry and material density, one can calculate the radii of gyration. For complicated parts, this can be time consuming. Often, the technician does not have access to the rotor's dimensional details in order to make the calculations. Hence, an inexpensive empirical technique such as the one described is valuable.

The procedure makes use of measuring the natural frequency and key dimensions of a rotational pendulum formed by hanging the rotor on wires which have negligible mass. From these measured parameters, the radii of gyration are computed using the simple natural frequency formula described below.

A typical rotational pendulum arrangement for measuring the polar radius of gyration is shown in Figure 1a. Typical arrangements for measuring the diametric radius of gyration are shown in Figures 1b and 1c. Either diametric arrangement (Figures 1b or 1c) can be used. The choice is a matter of convenience depending upon the rotor configuration. To calculate high-speed rotor behavior, it is necessary to know both the polar and diametric radii of gyration.

In all cases, the pendulum wires should be parallel, have equal length L, be equidistant r from the rotor center of gravity and should share the rotor weight equally. Often, the center of gravity can be determined by geometric symmetry or by balancing the rotor on a knife edge or from a wire. The lengths L and r should be chosen such that errors in measuring length are negligible and that the pendulum natural frequency oscillations can be timed with a stop watch. Usually oscillations with natural frequency in the 0.2 to 2 Hz range can easily be counted and timed.

Pendulum oscillations are started with an initial rotational displacement (up to 10° single amplitude) then released such that the small-angle linearization assumption is valid. Since damping is usually small, tens of oscillations of approximately equal amplitude can be counted and timed giving an accurate and repeatable estimate of pendulum natural frequency.

The measured natural frequency f in Hz is determined as:

$$f = \frac{n}{t}$$

where: n = number of cycles
t = time for n cycles, sec.

The linearized differential equation of motion for the rotational pendulum is given by

$$\frac{d^2\theta}{dt^2} + \frac{r^2 g}{k^2 L}\,\theta = 0$$

where: θ = angular displacement the wire makes with the ver tical, rad
L = length of pendulum wire, in.
r = distance of the rotor center of gravity to the wire, in.
g = local gravitational acceleration, (~ 386.4 in./sec^2)
k = radius of gyration, in.

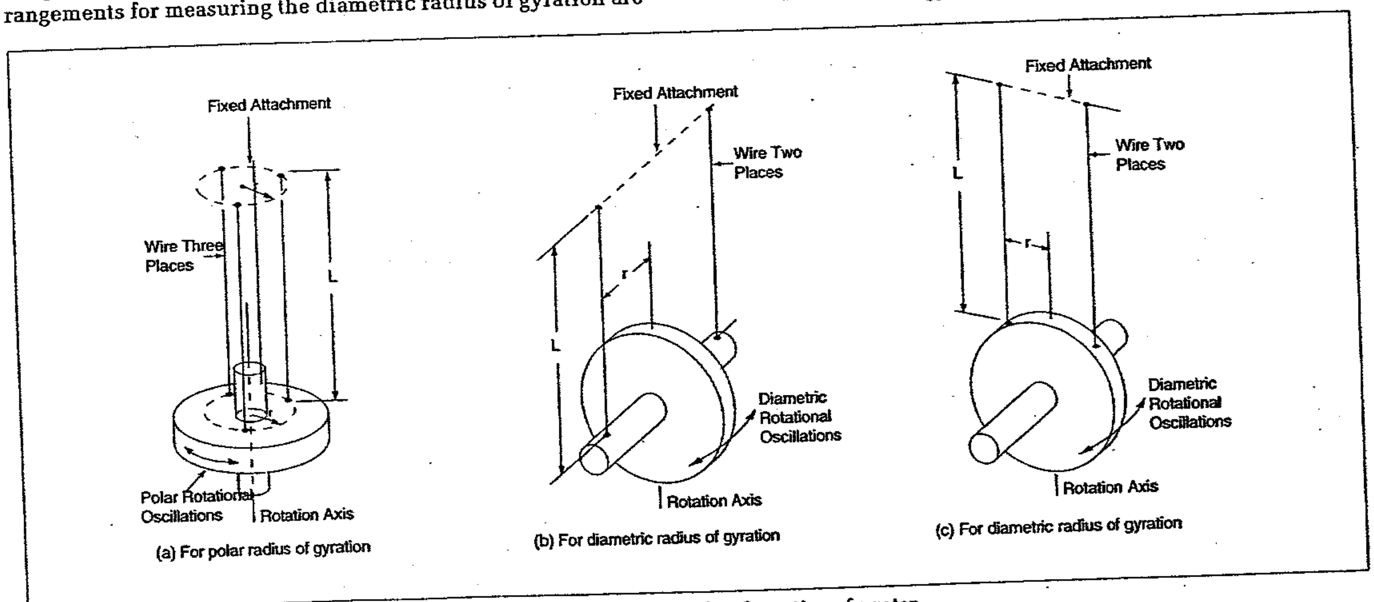

Figure 1. Typical rotational pendulum arrangements for measuring the radii of gyration of a rotor.

The following example is a practical application of the rotational pendulum method for measuring the mass moments of inertia in the field or laboratory.

A customer requests a spin test laboratory to proof-spin to 16,000 RPM a custom-made rotor for a radial-gap permanent magnet (PM) electric motor. The test purpose is to demonstrate that a hoop-wound fiberglass composit g overwrap will safely support the permanent ma s mounted on the outside diameter of a 6-in.-diameter by 2-in. thick steel hub. The hub has a 1-in. slightly tapered bore for a shaft that was unavailable for this test. An adapting arbor had to be fabricated for the spin test to support the 5.9 lb rotor assembly on a vertical cantilever quill shaft of an air turbine. The addition of the arbor adaptor reduced the ratio of polar-to-diametric mass moments of inertia of the assembly from 1.74 (without the adaptor) to 1.17 (with adaptor) as measured by the described pendulum method. The reduced ratio alerted test personnel that the rotor assembly has become more susceptible to disturbances and imbalances that can induce whirl. As a guideline for rigid rotor spin stability especially at speeds above the first rigid body mode, it is desirable to have the ratio of polar-to-diametric inertias of the rotor/adaptor assembly less than 0.8 (long cylinder-like) or greater than 1.2 (disk-like). The described method to measure mass moments of inertia is a quick way for the technician to check if the polar-to-diametric inertia ratio falls into the 0.8 to 1.2 warning range so that appropriate test precaution or test fixture redesign can be considered. In this rotor example where the 1.17 ratio is just inside the caution range, testing was pursued cautiously and asynchronous whirl was experienced above 12,000 RPM well above the 2000 RPM first rigid body mode. Fortunately, the whirl amplitude was low enough to complete the test without rebalance or adaptor redesign.

The solution to the differential equation yields th tu frequency f in Hz of the rotational pendulum as

$$f = \frac{1}{2\pi}\sqrt{\frac{r^2 g}{k^2 L}}$$

from which the radius of gyration can be calculated as

$$k = \frac{r}{2\pi f}\sqrt{\frac{g}{L}}$$

The mass moment of inertia I is obtained from the measu rotor weight w, using the classical definition of mass mom of inertia.

$$I = \frac{w}{g}k^2$$

As an example, for the measured parameters $L = 30$ in., r in., $n = 10$, $t = 10$ sec and $w = 10$ lb, the natural frequency f Hz, the radius of gyration $k = 2.856$ in., and the mass mom of inertia $I = 0.211$ lb-in.-sec^2. A time saver – to ensure reas able oscillation count and timing for a specific rotor, estim a value for radius of gyration k and use the frequency equat to determine trial values for lengths L and r before initiat the test setup.

In summary, the rotational-pendulum technique is an in pensive and accurate method to determine radii of gyratio rotors. The advantages of the rotational pendulum method a) no rotor-specific adapters or fixtures, b) no special ins mentation, c) quick and simple setup and measurements, d) good accuracy depending upon length and frequency m surements. The method is also applicable to other comp structures as long as they can be appropriately susp ed rotational pendulum.

Torsional Vibration

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Torsional vibration in rotating machinery components and a procedure for calculating torsional vibration forces are reviewed. A machinery failure case history is used to illustrate the techniques.

Torsional vibration is a form of vibration that is generally difficult to conceptualize. This condition is further complicated by the fact that problems associated with torsional vibration usually occur on rotating systems which are difficult to instrument and to observe firsthand. Often, the first indication of a problem is that a shaft on a new piece of equipment breaks for no apparent reason. A subsequent analysis of the system indicates that the design of the shaft is well within allowable limits for the loading on the system but when the system is put back into service, the shaft breaks again in about the same period of time. Such an event occurred on a 700 horsepower induced draft fan. The fan was part of a utility power boiler system. The fan system consisted of a motor, a variable speed magnetic clutch, and the fan (see Figure 1). There were identical couplings between the motor and the clutch and the clutch and the fan. When the fan was operated at full load, the coupling between the clutch and the fan would fail after approximately 200 hours of use, but the identical coupling between the motor and the clutch operated without incident. After a somewhat exhaustive investigation, it was learned that there was a torsional vibration problem between the clutch and the fan and this is why the coupling failed. This incident will be discussed in greater depth later in this article.

Basic Concepts

Torsional vibration is the tendency of one end of a system to rotate relative to the other end of the system about a common axis due to torque in the system and torsional inertia. A child spinning on a swing is one example of such a system. The child sits on the swing and twists himself up in the ropes until he is sure he will get good and dizzy. He then lifts his feet off the ground and begins to spin. He spins and spins until the ropes are straight again. At this point the torque that started the system rotating is zero, but the child continues to rotate due to inertia. He continues to rotate until the kinetic energy of his rotating inertia is converted into potential energy stored in the torque on the ropes. He then stops momentarily and begins to spin the other way. If the child is not stopped, this motion of rotating first in one direction and then in the other will continue until all the potential energy originally put into the system is dissipated by damping forces. However, if a relatively small push is given at the right time during each cycle, this motion can continue indefinitely.

The preceding simple example of torsional vibration was made for illustrative purposes. Systems of more practical interest might be a steam turbine and generator, the power train of an automobile or a large structure that twists when the wind blows a certain way. The following is a brief theoretical discussion of torsional vibration.

The basic equation of undamped torsional vibration is the same as that of any other form of vibration:

$$\mu = \sqrt{K_{eff}/M_{eff}}$$

where: μ = the natural frequency of the system in radians per second
K_{eff} = the effective spring constant of the system
M_{eff} = the effective mass of the system.

The effective spring constant and mass of torsional systems are somewhat different from those of lateral systems. The effective spring constant is the torsional rate or the amount of

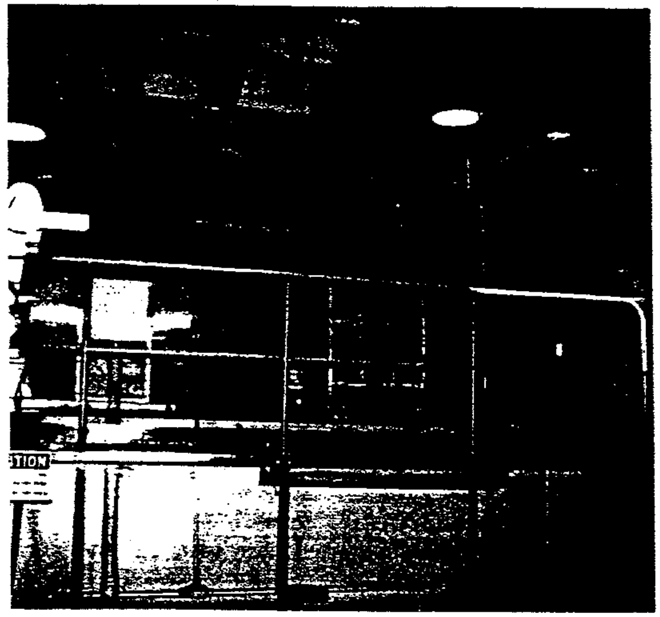

Figure 1. Induced draft fan showing motor, magnetic clutch and fan.

twist that a given torque will impart on the system. The numerical value of the torsional spring constant can be calculated as follows:

$$K_{eff} = GJ/L$$

where: G = the shear modulus of the material
J = the polar moment of inertia of the cross section
L = the axial length of the member

Values of shear modulus for several engineering metals are given in Table 1.

The value of the polar moment of inertia J of circular or circular tube members can be determined by the following equation:

$$J = (\pi c^4/2) - (\pi b^4/2)$$

where c = the outside radius and b = the inside radius. The determination of the torsional stiffness of non circular members is more complicated. Good treatments on this subject can be found in Chapter 10 of *Timoshenko* and Chapter 6 of *Ugural*. Useful approximations of the torsional stiffness of many cross-sectional shapes can be found in Chapter 9 of *Roark*.

The determination of the effective spring constant of structures is the sum of the contributions of the individual members plus the structural stiffness. The torsional stiffness of the individual members can be readily determined with the aid of the references mentioned above. The torsional stiffness of all the members are then summed. The second part of this type of

Table 1. Shear modulus of selected engineering metals.

Metal	Shear modulus × 10^6 lb/in^2
Aluminum Alloys	3.7-3.9
Brass	5.3-6.0
Bronze	5.1-5.9
Copper	5.8
Inconel	11.0
Iron, Malleable	9.3
Iron, Cast	5.2-8.2
Monel	9.5
Steel	11.0-11.9
Stainless Steel, 18-8	10.6

problem is the determination of the contribution of the structure itself. The structural members that hold the system together can bend and be in compression or tension as well as twist, as the system vibrates torsionally. The components of all of these forces that contribute to the torsional stiffness of the system must be taken into account in order to determine the effective torsional stiffness of the entire structure. The structural torsional stiffness of the system is determined by summing the contributions of each of the members. The first step in this process is to determine the centroidal axis of the structure. This is the line about which the system rotates. It is located as follows:

1. Determine the relative stiffness of each of the members as it distorts in the direction of rotation.
2. Make a diagram of the cross section of the structure and indicate the location of the legs.
3. Assign an arbitrary set of cartesian coordinates to the cross section. The problem can be simplified by choosing one of the members as the origin and having the X axis go through one of the other legs.
4. Multiply the relative stiffness of each member by its distance along the X axis and sum all of these values.
5. Sum the relative stiffness of all the legs.
6. Divide the quantity obtained in step 4 by the quantity obtained in step 5. This distance is the X coordinate of the centroidal axis.
7. Repeat this process along the Y axis to obtain the Y value.

Once the centroidal axis is located, the torque created by each of the members as the system is distorted torsionally can be determined. This is done by multiplying the relative stiffness of each member by its distance from the centroidal axis. These values are then summed. This value is the structural torsional stiffness of the system. It is then added to the torsional stiffness of the individual members determined earlier. This combined value is the effective torsional spring constant of the system.

The effective mass of a torsional system is the mass moment of inertia. If the axis of rotation goes through the centroid of the system, the mass moment of inertia I is the mass M of the system times the radius of gyration squared R^2:

$$I = R^2M$$

The quantity WR^2 for a rotor should be indicated on the print of that rotor. This quantity is the weight of the rotor times the radius of gyration squared. It is important that this quantity be known for all rotors and not just the driven rotors. The reason for this will be made clear later in this article. The mass can be determined by dividing the weight by gravitational acceleration (386 in/sec^2). The radius of gyration is the distance from the centroidal axis to a distance at which all the mass of the system can be considered to be concentrated. This value for most engineering shapes can be found in handbooks such as *Roark*. For the hardy souls, the mass moment of inertia can be determined directly by integrating the radius of gyration squared times the differential of mass:

$$I = \int_v R^2 dM$$

This integral is performed over the entire volume of the effective mass of the system.

If the axis of rotation does not pass through the centroid of the system, another variable is introduced. In such a system, the mass moment of inertia I is the mass moment of inertia of the system about its centroid I_c plus the mass of the system M times the distance squared d^2 between the centroidal axis and the axis of rotation:

$$I = I_c + Md^2$$

In systems in which the distance d is very large in relation to the radius of gyration R, the effects of rotary inertia of the mass about its own centroid will usually be negligible compared with the mass moment of inertia of the system about the axis of rotation. An example of such a system is a swinging pendulum.

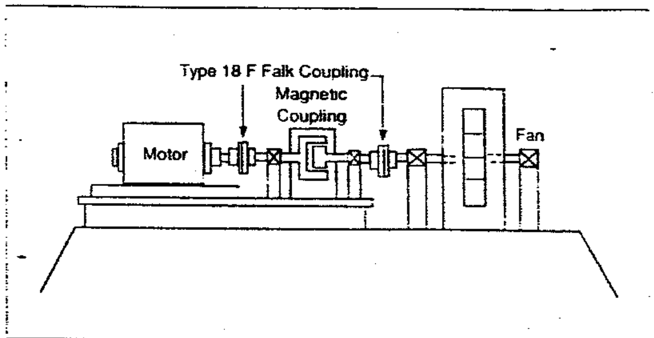

Figure 2. Motor, magnetic coupling and fan configuration.

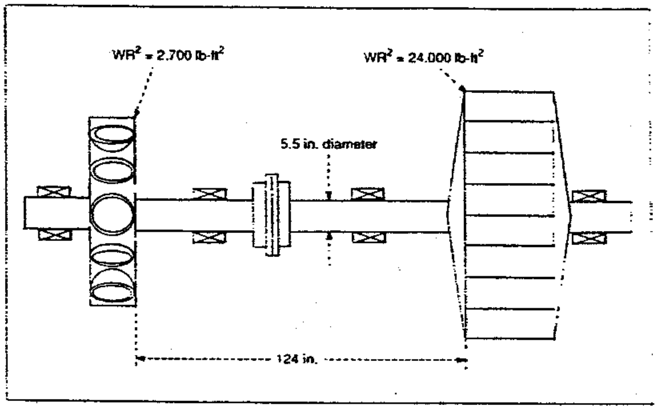

Figure 3. Fan side system with inertial masses and torsional members.

Excitation Forces

The calculation of torsional resonant frequency has been discussed so far, but nothing has been said about the excitati forces. These forces are generally more complicated tha. those associated with lateral vibration. For example, in rotating equipment the dynamic forces associated with running speed such as unbalance, which are of great importance in lateral vibration, usually have little to do with torsional vibration. The reason for this is that torsional vibration is caused by excitation and inertial forces within the rotating assembly, and the speed with which this assembly is rotating relative to stationary surroundings has no direct effect on the rotating assembly. Rotating speed will have an indirect effect if there is an interaction between the rotating assembly and the stationary surroundings. For example, a two-cycle internal combustion engine will put out a pulse every time a cylinder fires. If this engine has two cylinders and is driving an electric generator at 3600 rpm, it will put out a pulsating torsional impulse at a frequency equal to the running speed times the number of cylinders (7200 cpm/120 Hz).

Another example of a torsional excitation force in electrically driven equipment is the relationship between slip speed and the rotor bars. As the electrically driven rotor slips behind the field rotating speed, electric current is induced in the bars and shorting rings. This current creates a magnetic field in the iron. Every time a pair of rotor bars is cut by the rotating field, a pulse occurs. The frequency of this excitation force is the slip speed times the number of pairs of rotor bars.

Failure Case History

Returning to the case of the induced draft fan mentioned earlier, Figure 2 is a diagram of the system. It consisted of an 870 rpm, 700 hp induction motor, a variable speed magnetic clutch, and 190,000 CFM fan. There are Falk Steelflex 18F cc plings between the motor and the magnetic clutch and t clutch and the fan. Both couplings were under the same torque. The motor side coupling remained in service without incident, but the fan side coupling would fail after approximately 200 hours of service. The system from the fan side of the clutch to the fan was examined (see Figure 3). This part of the system consisted of two significant inertial masses (the fan rotor and the fan side clutch rotor) and two springs (the shaft and the coupling). When springs are in "series" such as in this incident, the effective spring constant of the entire system is the reciprocal sum of the individual components. Likewise when masses are attached to opposite ends of a spring with no other constraints, the effective mass of the system will be the reciprocal sum of the individual effective masses. That is, the effective mass of the entire fan side system is the sum of the effective mass of the fan side clutch rotor plus the effective mass of the fan rotor. It is assumed that the effective mass of the shaft is negligible.

$$1/M_t = (1/M_c) + (1/M_f)$$

Similarly, the effective spring constant of the entire fan side spring constant is the sum of the effective spring constant of the shaft plus the effective spring constant of the coupling.

$$1/K_t = (1/K_s) + (1/K_c)$$

The values of WR^2 for the two rotors were obtained from equipment technical data and converted to appropriate units:
$M_c = 1006$ lb-in-sec^2
$M_f = 8960$ lb-in-sec^2
$M_t = 904$ lb-in-sec^2
The spring constant of the coupling was determined experimentally, and the spring constant of the shaft was calculated:
$K_c = 8 \times 10^6$ lb-in/rad
$K_s = GJ/L = 8.35 \times 10^6$ lb-in/rad
[G(1035 steel) $= 11.5 \times 10^6$ lb/in^2,
$j = \pi D^4/32$ in^4, $L = 124$ in]
$K_t = 4.1 \times 10^6$ lb-in/rad
With this information, the resonant frequency of the system could be calculated:

$$\mu = \sqrt{K_{eff}/M_{eff}} = 67.3 \text{ rad/sec} = 10.7 \text{ cycles/sec}$$

This value was compared with the pulse frequency created by clutch slip. There were 24 sets of bars in the clutch. At full load, clutch slip speed was approximately 27 rpm. Pulse frequency was then calculated:

$$F_P = 24(27/60) = 10.8 \text{ pulses/sec}$$

This value is for all practical purposes identical to the system resonant frequency. The problem was addressed by reworking the magnetic clutch so that there were 29 sets of rotor bars. Subsequent tests have determined that the stress on the fan side shaft was reduced by a factor of 38.

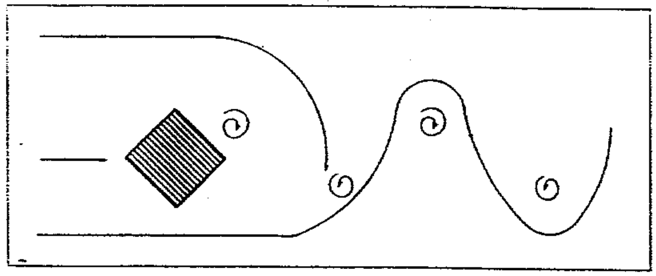

Figure 4. Vortices being shed at regular intervals from alternating sides of a structure in a flow path.

Structural Vibration

Torsional vibration associated with buildings, bridges and some other structures is often caused by vortex shedding. Objects in a flow stream such as wind will shed vortices at regular intervals. These vortices will be shed from one side and then the other (see Figure 4). This action will cause an alternating torsional moment on the structure. If the frequency at which vortices are shed matches a natural frequency of the structure, a resonant frequency problem will probably develop. The frequency at which vortices will be shed can be determined with the following formula:

$$f_S = SU/D$$

where f_S = the vortex shedding frequency
S = the Strouhal number
U = the free stream velocity
D = the projected diameter to the stream flow
The Strouhal number for a variety of shapes can be found in Chapter 3 of *Blevins.*

In many systems, the point at which the spring stops and the effective mass begins is not clearly evident. Some systems have to be treated as continuous systems in which the unit mass moment of inertia and spring constant must be integrated over the length of the structure in order to determine the natural frequencies. In continuous systems, there is theoretically an infinite number of natural frequencies, but in actual systems the resonant frequencies of the higher modes is usually so high that they are not of practical importance.

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