Consider the angle ψ between the radius vector and the tangent line to a curve, $r = f(\theta)$, given in polar coordinates, as shown in Fig. 1. Show that $\psi = \tan^{-1}(r/(dr/d\theta))$.

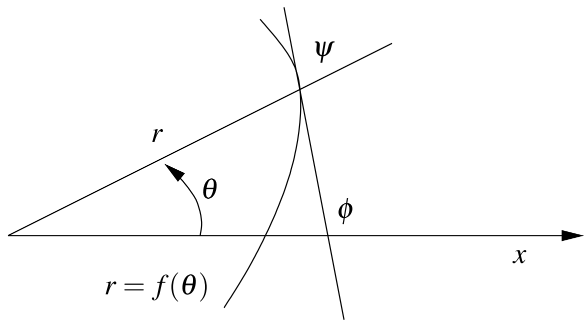

Figure 1: The tangent line to the curve $r = f(\theta)$ makes an angle of ψ with respect to the radial line at the point of tangency, and an angle ϕ with respect to the x-axis.

Proof:

- Consider $\phi = \theta + \psi$. Then $r = f(\theta)$ is given in polar coordinates by

$$x = r\cos\theta, \qquad y = r\sin\theta, \tag{1}$$

  with associated derivatives given by.

$$\frac{dx}{d\theta} = -r\sin\theta + \cos\theta\frac{dr}{d\theta}. \qquad \frac{dy}{d\theta} = r\cos\theta + \sin\theta\frac{dr}{d\theta}. \tag{2}$$

- From the geometry of the problem in which it is evident that $2\pi - \psi - \theta = 2\pi - \phi$, we have that $\psi = \phi - \theta$, and consequently, using a familiar multiple angle formula from trigonometry, that

$$\tan\psi = \tan(\phi - \theta) = \frac{\tan\phi - \tan\theta}{1 + \tan\phi\tan\theta}. \tag{3}$$

- Since the tangent line to the curve $f(\theta)$ makes an angle ϕ with respect to the $x-$axis we have that $\tan\phi = \dfrac{dy/d\theta}{dx/d\theta}$, and trivially from the geometry that $\tan\theta = y/x$. Substituting these into (3) gives,

$$\tan\psi = \frac{\dfrac{dy/d\theta}{dx/d\theta} - \dfrac{y}{x}}{1 + \left(\dfrac{y}{x}\right)\dfrac{dy/d\theta}{dx/d\theta}} = \frac{x\dfrac{dy}{d\theta} - y\dfrac{dx}{d\theta}}{x\dfrac{dx}{d\theta} + y\dfrac{dy}{d\theta}} \tag{4}$$

$$= \frac{x\left(r\cos\theta + \sin\theta\dfrac{dr}{d\theta}\right) - y\left(-r\sin\theta + \cos\theta\dfrac{dr}{d\theta}\right)}{x\left(-r\sin\theta + \cos\theta\dfrac{dr}{d\theta}\right) + y\left(r\cos\theta + \sin\theta\dfrac{dr}{d\theta}\right)} \tag{5}$$

Substituting (1) into (5) yields,

$$\tan\psi = \frac{r\cos\theta\,(r\cos\theta+\sin\theta(dr/d\theta)) - r\sin\theta\,(-r\sin\theta+\cos\theta(dr/d\theta))}{r\cos\theta\,(-r\sin\theta+\cos\theta(dr/d\theta)) + r\sin\theta\,(r\cos\theta+\sin\theta(dr/d\theta))} \quad (6)$$

$$= \frac{r^2}{rdr/d\theta} \quad (7)$$

$$= \frac{r}{dr/d\theta} \quad (8)$$

which is the desired result. □