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A Bulk-Flow Theory for Turbulence in Lubricant Films

The purpose of this study on the turbulent lubricant film is:

1 To give a brief outline of a new theory called bulk-flow theory;
2 To investigate to what extent results of theories based on law of wall and mixing length concept agree with the newly developed theory;
3 To provide a theoretical basis for the design of bearings lubricated by fluids of low kinematic viscosity.

Introduction

THE main characteristic of the bulk-flow theory for the turbulent lubricant film is the fact that it does not explicitly make use of any information, nor of any model, on

1 fluctuations of local velocities of flow due to turbulence;
2 the shape of flow velocity profiles from which fluctuating components have been eliminated through averaging.

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The theory is entirely based on bulk-flow. This is in contrast with previously developed theories by Constantinescu [5][1] who uses the mixing length model, Ng and Pan [10] and Elrod and Ng [8] who use the law of wall and an eddy-viscosity concept and Burton [4] and Black [1] who use information on the shape of the time-averaged flow velocity profile in the lubricant film.

In this theory only the bulk-flow relative to a surface or wall and the corresponding shear stress at that surface or wall under a given set of conditions of turbulent flow are considered and correlated, see Hirs [9].

This approach is essentially a logical extension to basic work done by Blasius [2] on turbulent "pressure flow," i.e., under the influence of a pressure gradient in a pipe, by Davies and White [7] on a similar flow between two stationary parallel surfaces, by Couette [6] on turbulent "drag flow" between two concentric

[1] Numbers in brackets designate References at end of paper.

Nomenclature

a = weighting factor
$G_x G_y$ = constants defined in equations (11*a*) and (11*b*)
h = film thickness
$m; m_0; m_1$ = constants, see formulas (1), (2), and (3)
$n; n_0; n_1$ = constants, see formulas (1), (2), and (3)
p = pressure
p_1 = fictitious pressure
p_r = representative pressure
$R = \frac{\rho U h}{\eta}$ = Reynolds number
U = sliding velocity
u_1, u_2 = surface velocities with respect to the film
u_m = mean velocity of flow relative to stationary surface
u_x = ditto in x direction
u_y = ditto in y direction
u_s = ditto in s direction
$U_x = \frac{u_x}{U}$ = dimensionless mean velocity of flow
$U_y = \frac{u_y}{U}$ = dimensionless mean velocity of flow
$U_1 = \frac{u_1}{U}$ = dimensionless surface velocity
$U_2 = \frac{u_2}{U}$ = dimensionless surface velocity
s = common coordinate of resultant flow and resultant pressure gradient
t = time
$x, y,$ = coordinates in sliding direction and at right angles to sliding direction attached to stationary surface
$\mathbf{x}, \mathbf{y},$ = coordinates in sliding direction and at right angles to sliding direc-

(Continued on next page)

cylindrical surfaces due to the sliding of one surface, and by investigators who later have added experimental results of related types of flows to the previous pioneer work. Briefly summarized, the present theory is primarily based on the empirical finding that the relationship between wall-shear stress and mean velocity of flow relative to the wall at which the shear stress is exerted can be expressed by a common, simple formula for pressure flow, for drag flow, and for combinations of these two basic types of flow:

$$\frac{\tau}{{}^1/_2\rho u_m{}^2} = n\left(\frac{\rho u_m h}{\eta}\right)^m \qquad (1a)$$

where

- τ = wall-shear stress
- ρ = density of fluid
- η = dynamic viscosity of fluid
- u_m = mean velocity of flow relative to wall or surface at which shear stress τ is exerted
- h = film thickness
- n and m = empirical numerical constants to be fitted to the available experimental results
- $\frac{\rho u_m h}{\eta}$ = Reynolds number
- $\frac{\tau}{{}^1/_2\rho u_m{}^2}$ = friction factor

Values for n and m fitted to individual experiments can be shown to depend, albeit rather weakly, on:

1 the roughness of the surfaces;
2 the curvature of the surfaces;
3 the question of whether we have to do with Reynolds numbers greater than 100,000;
4 the influence of inertia effects other than those inherent in turbulence in the flow;
5 the type of flow:
   (a) "pressure flow" under the influence of a pressure gradient
   (b) "drag flow" due to the sliding of a surface
   (c) the nature of the combination, if any, of both types of flow;
6 the rates of change with time and from place to place of all quantities indicated in formula (1a); these quantities may vary moderately in a lubricant film without violating the applicability of formula (1a).

It is stressed that mean flow velocity u_m in formula (1a) is taken relative to the surface at which shear stress τ is exerted, while bearings have two surfaces at which shear stresses are exerted. Therefore, more specialized formulas than (1a) are presented for either of the two surfaces. It is assumed that the frame of reference is attached to a surface that is considered stationary. Mean flow velocity u_m in the film and sliding speed U of the sliding surface are taken with respect to the frame of reference in the same direction. Thus the treatment is restricted to unidirectional flow. So, two formulas can now be derived:

one for the stationary surface

$$\frac{\tau_a}{{}^1/_2\rho u_m{}^2} = n\left(\frac{\rho u_m h}{\eta}\right)^m \qquad (1b)^2$$

and another for the sliding surface

$$\frac{\tau_b}{{}^1/_2\rho(u_m - U)^2} = n\left(\frac{\rho(u_m - U)h}{\eta}\right)^m \qquad (1c)^2$$

in which the wall shear stresses are characterized by subscripts a and b, respectively.

Formulas (1a), (1b), and (1c) serve for developing a bulk-flow theory. Formula (1a) appears to be valid for both pressure flow and drag flow within limits, compare Couette [6] and Davies and White [7] (see also Burton [3]). Formulas (1b) and (1c) clarify the feature that, in a lubricant film, there are shear stresses at either surface (τ_a and τ_b) and mean flow velocities with respect to either surface (u_m and $u_m - U$).

A Comparison Between Pressure Flow and Drag Flow

For our brief outline of the bulk-flow theory it is useful to realize that similarity of the two types of flow (flow under the influence of a pressure gradient and flow due to the sliding of a surface) is not only evident from the fact that the two relationships for τ have a similar form but also from the fact that the two values for n as well as the two values for m differ but little. This similarity can be further clarified by considering the two extreme cases represented by formulas (1b) and (1c) as far as the type of flow is concerned.

(a) solely flow under the influence of a pressure gradient, pressure flow (Fig. 1)

$$\frac{\tau_0}{{}^1/_2\rho u_{m_0}{}^2} = n_0\left(\frac{\rho u_{m_0} h}{\eta}\right)^{m_0} \qquad (2)$$

which can be derived from (1b) and (1c) by inserting $U = 0$ and which gives equal shear stresses on the two surfaces: $\tau_a = \tau_b = \tau_0$

(b) solely flow due to the sliding of a surface, drag flow (Fig. 2)

$$\frac{\tau_1}{{}^1/_2\rho u_{m_1}{}^2} = n_1\left(\frac{\rho u_{m_1} h}{\eta}\right)^{m_1} \qquad (3)$$

which is also based on (1b) and (1c) and in which $u_m = {}^1/_2U$ foe the stationary surface and $u_m = -{}^1/_2U$ for the sliding surfacr and which should be taken to yield equal but opposite shear stresses on the two surfaces: $\tau_a = -\tau_b = \tau_1$.

[2] For physical reasons, the mth power must be treated as if it were an odd number in order to make the functional relationship uneven.

Nomenclature

tion attached to the body of the film

- z = coordinate perpendicular to x and y
- η = dynamic viscosity of flowing fluid
- ρ = density of flowing fluid
- $\frac{\rho u_m h}{\eta}$ = Reynolds number
- τ = shear stress at a surface
- $\frac{\tau}{{}^1/_2 u_m{}^2}$ = friction factor
- τ_0 = shear stress at a surface due to flow under the influence of a pressure gradient
- τ_1 = ditto due to the sliding of a surface
- τ_a = shear stress at stationary surface
- τ_b = shear stress at sliding surface
- $\frac{\partial P}{\partial X} = \frac{\rho h^3}{\eta^2}\frac{\partial p}{\partial x}$ = dimensionless pressure gradient
- $\frac{\partial P}{\partial Y} = \frac{\rho h^3}{\eta^2}\frac{\partial p}{\partial y}$ = dimensionless pressure gradient
- $\text{Grad } P = \left[\left(\frac{\partial P}{\partial X}\right)^2 + \left(\frac{\partial P}{\partial Y}\right)^2\right]^{1/2}$ = resultant dimensionless pressure gradient

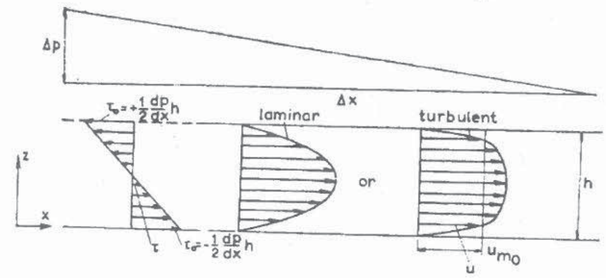

Fig. 1 Pressure flow between two surfaces under the influence of a pressure gradient

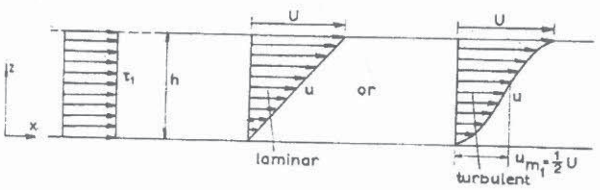

Fig. 2 Drag flow between two surfaces due to the sliding of one of them

The two extreme cases represented by formulas (2) and (3) show the above-mentioned characteristic, which is derived from experiments mentioned above, viz., the value of n_0/n_1 as well as that of m_0/m_1 is close to unity. This similarity of formulas (2) and (3) exists despite the differences in shear stress distribution between:

1 pressure flow, see Fig. 1, with an inversely symmetrical shear stress distribution varying linearly with height z;

2 drag flow due to the sliding of a surface, see Fig. 2, with a constant shear stress distribution, independent of height z;

The absence of a predominant influence of the shear stress profile across the film on the values of constants n and m in the formulas (2) and (3) leads to the following conclusions:

1 The relationship between the shear stress at a surface and the mean velocity of flow relative to that surface depends but little on the type of flow, i.e., being valid to a reasonable approximation for pressure flow, drag flow as well as with a combination of both, see Fig. 3.

2 It is tempting to entirely neglect this sensitivity of the shear stress at a surface to the type of flow. Such a neglect entails that one might ascribe an additive nature to the shear stresses at the two surfaces given in formulas (1*b*) and (1*c*), i.e., in that the total shear stress at a surface may be found by summing the two component shear stresses as shown in Fig. 3: $\tau_a = \tau_0 + \tau_1$ for the stationary surface and $\tau_b = \tau_0 - \tau_1$ for the sliding one.

However, in a more refined treatment it should be taken into account that, for one and the same average flow velocity, density, viscosity, and film thickness in either extreme case represented by formulas (2) or (3), wall-shear stress for one type of flow is not exactly equivalent with wall-shear stress for the other type of flow. In fact, the two shear stresses, although not differing too much, show a ratio different from unity. This ratio may be assessed by dividing formula (2) by formula (3) and assuming $u_{m_0} = u_{m_1}$ as well as identical p, η, h for the two cases, and replacing u_{m0} and u_{m1} by their common value u_m, viz.,

$$\frac{\tau_0}{\tau_1} = \frac{n_0}{n_1}\left(\frac{\rho u_m h}{\eta}\right)^{m_0 - m_1} \qquad (4a)$$

where suffix 0 stands again for pressure flow and suffix 1 for drag flow. Experimental results show m_0 and m_1 to be equal within the measuring error and show n_0 to be maximum twenty percent greater than n_1, so that

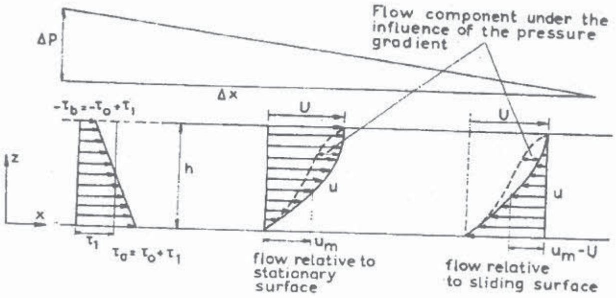

Fig. 3 Turbulent flow between two surfaces under the influence of a pressure gradient and due to the sliding of a surface

$$\frac{\tau_0}{\tau_1} = a \approx 1.2 \qquad (4b)$$

in which a will be used as a weighting factor with a magnitude close to unity.

Now, in expressions for τ_a and τ_b (see formulas (1*b*) and (1*c*)) one of the components can be weighted, say τ_1. Thus, $\tau_a = \tau_0 + a\tau_1$ and $\tau_b = \tau_0 - a\tau_1$. The significance of introducing weighting factor a is the fact that shear stresses τ_a and τ_b can be said to be due to just one type of flow. Equations for τ_a and τ_b need not now be any more subject to the nature of the combination of the two types of flow.

Equations for Unidirectional Flow

Since in lubricant films the pressure build-up is of major importance, the description of two types of flow will henceforth be simplified to the description of only one type of flow, namely the flow under the influence of a representative "total" pressure gradient

$$\left(\frac{dp_r}{dx}\right),$$

in which the influence of the drag flow component on the shear stress is included. That is, in order to account for the occurrence of the drag flow component, a fictitious pressure gradient

$$\left(\frac{dp_1}{dx}\right)$$

will be introduced and it will be added to the actual pressure gradient

$$\left(\frac{dp}{dx}\right)$$

so as to obtain the representative gradient. Weighting factor a in formula (4*b*) will be used when converting the shear stresses τ_1, which are due to the drag flow component to the shear stresses ($a\tau_1$) that are ascribed to a fictitious pressure flow component, characterized by a fictitious pressure gradient

$$\left(\frac{dp_1}{dx}\right).$$

For the pressure flow in the steady state to which we will here confine ourselves, it follows from the equilibrium between the shear stress τ_0 acting on elements dx on the two surfaces and the actual pressures p and $p + dp$ acting on film thickness h that $2\tau_0 dx = ph - (p + dp)h$. Thus,

$$\tau_0 = -{}^1/_2 \frac{dp}{dx} h \qquad (5a)$$

In analogy, the following relationship is introduced for defining the fictitious pressure gradient $\left(\frac{dp_1}{dx}\right)$ which is henceforth taken representative of drag flow

$$a\tau_1 = -{}^1/_2 \frac{dp_1}{dx} h, \quad \text{so that } \frac{dp_1}{dx} = -2a\frac{\tau_1}{h} \tag{5b}$$

At the stationary surface we then have

$$\tau_0 + a\tau_1 = -\frac{h}{2}\frac{d}{dx}(p + p_1);$$

which does contain the representative pressure gradient $\frac{d(p + p_1)}{dx} = \frac{dp_{ra}}{dx}$ generating flow with average flow velocity u_m relative to the stationary surface.

It follows from formulas (1) and (2) for the stationary surface that

$$\frac{-h\frac{d}{dx}(p + p_1)}{\rho u_m^2} = n_0\left(\frac{\rho u_m h}{\eta}\right)^{m_0} \tag{6a}$$

At the sliding surface we may put:

$$\tau_0 - a\tau_1 = -\frac{h}{2}\frac{d}{dx}(p - p_1).$$

This gives the representative pressure gradient $\frac{d(p - p_1)}{dx} = \frac{dp_{rb}}{dx}$ generating flow with average flow velocity $(u_m - U)$ relative to the sliding surface (in which U sliding speed).

It follows for the sliding surface from formulas (1) and (2)

$$\frac{-h\frac{d}{dx}(p - p_1)}{\rho(u_m - U)^2} = n_0\left(\frac{\rho(u_m - U)h}{\eta}\right)^{m_0} \tag{6b}$$

The fictitious pressure gradient can be eliminated from these two formulas (6a) and (6b). The result is a formula in which only the actual pressure gradient is left, and thus can be determined for any combination of pressure flow and drag flow, provided that average velocity u_m and sliding speed U have parallel directions:

$$\frac{dp}{dx} = -{}^1/_2 n_0\left[\frac{\rho u_m^2}{h}\left(\frac{\rho u_m h}{\eta}\right)^{m_0} + \frac{\rho(u_m - U)^2}{h}\left(\frac{\rho(u_m - U)h}{\eta}\right)^{m_0}\right] \tag{6c}$$

If the inertia effects other than those inherent in the turbulence character of the flow are negligible, it is now possible to forthwith determine the pressure build-up and load-carrying capacity of bearings having no side leakage (which would result in cross flow) i.e. bearings having infinite width. It is remarkable that, by eliminating the fictitious pressure gradient, the magnitude of weighting factor a in formula (4b) does not affect in any way the magnitude of the actual pressure gradient in (6c).

One may also eliminate the actual pressure gradient from (6a) and (6b):

$$\frac{dp_1}{dx} = -{}^1/_2 n_0\left[\frac{\rho u_m^2}{h}\left(\frac{\rho u_m h}{\eta}\right)^{m_0} - \frac{\rho(u_m - U)^2}{h}\left(\frac{\rho(u_m - U)h}{\eta}\right)^{m_0}\right] \tag{6d}$$

Thereby obtaining the following expression for the fictitious pressure gradient

$$\frac{dp_1}{dx} = -2a\frac{\tau_1}{h} = -2\frac{n_0}{n_1}\frac{\tau_1}{h}, \text{ viz.,} \tag{5b}$$

In formula (5b) the importance of weighting factor a becomes evident and shear stress due to drag flow can be derived:

$$\tau_1 = {}^1/_4 n_1\left[\rho u_m^2\left(\frac{\rho u_m h}{\eta}\right)^{m_0} - \rho(u_m - U)^2\left(\frac{\rho(u_m - U)h}{\eta}\right)^{m_0}\right] \tag{6e}$$

Thus empirical constant n_1 appears to be tied up with expressions for shear stress due to sliding in the same way as previously shown in a less general formula (3). For that matter, empirical constant n_0 appears to be tied up with actual pressure gradients and shear stresses due to actual pressure gradients, see formula (6c) and the less general formula (2).

Equations for Flow in Mutually Perpendicular Directions

The previous method, and the use in it of a fictitious pressure gradient, in particular, will be seen to be useful in an extension of the theory to flow in a lubricant film in mutually perpendicular directions. The extension of the theory proves possible by making the following assumptions:

1 that a fictitious pressure component may be conceived in the lubricant film so as to account for the drag flow component;

2 that for the *stationary* surface a relation can be given between, on the one hand, the gradient of the representative pressure consisting of the actual pressure plus the additional fictitious pressure, and, on the other hand, the mean velocity of flow relative to the stationary surface, the density, the viscosity, and the film thickness, in accordance with equation (6a);

3 that for the *sliding* surface a relation can be given between, on the one hand, the gradient of the representative pressure consisting of actual pressure minus the fictitious pressure and on the other hand, the mean velocity of flow relative to the sliding surface, the density, the viscosity, and the film thickness, in accordance with equation (6b);

4 that the resultant representative gradient $\frac{dp_r}{ds}$ and the resultant mean velocity of flow u_s for one and the same surface have the same direction;

5 that such directions for stationary and sliding surface will not necessarily coincide, see Fig. 4.

These assumptions, and (4) and (5) in particular, lead to the following slightly generalized form of (6a) and (6b):

$$\frac{-h\frac{dp_r}{ds}}{\rho u_s^2} = n_0\left(\frac{\rho u_s h}{\eta}\right)^{m_0} \tag{6f}$$

where coordinate s and suffix s indicate the *common* direction of the resultant pressure gradient and the resultant mean flow velocity;

in which, for the stationary surface, the pressure gradient $\frac{dp_{ra}}{ds}$ is the vectorial resultant of components $\frac{\partial(p + p_1)}{\partial x}$ and $\frac{\partial(p + p_1)}{\partial y}$ and mean velocity u_s is the vectorial resultant of components u_x and $u_{rb} = u_y$;

in which, for the sliding surface, the pressure gradient $\frac{dp_{rb}}{ds}$ is the vectorial resultant of components

$$\frac{\partial(p - p_1)}{\partial x} \quad \text{and} \quad \frac{\partial(p - p_1)}{\partial y}$$

and mean velocity u_s is the vectorial resultant of flow components $(u_x - U)$ and u_y;
where the xy coordinate system lies in the plane of and is attached to the stationary surface;
where the y direction is at right angles to the sliding direction;
where x is the sliding direction;
and where u_x and u_y are the mean velocities of flow relative to the x and y-directions, respectively.
Thus, equations for the stationary and the sliding surface can be derived:

1 Stationary Surface. By suitably resolving the resultant representative pressure gradient

$$\frac{dp_{rs}}{ds} = \frac{d(p + p_1)}{ds}$$

and the resultant mean flow velocity

$$u_s = (u_x^2 + u_y^2)^{1/2}$$

in x and y-directions, it follows from formula (6f)

$$\frac{-h\frac{\partial}{\partial y}(p + p_1)}{\rho u_x(u_x^2 + u_y^2)^{1/2}} = n_0\left\{\frac{\rho(u_x^2 + u_y^2)^{1/2}h}{\eta}\right\}^{m_0} \qquad (7a)$$

$$\frac{-h\frac{\partial}{\partial y}(p + p_1)}{\rho u_y(u_x^2 + u_y^2)^{1/2}} = n_0\left\{\frac{\rho(u_x^2 + u_y^2)^{1/2}h}{\eta}\right\}^{m_0} \qquad (7b)$$

2 Sliding Surface. By suitably resolving resultant gradient $\frac{dp_{rb}}{ds} = \frac{d(p - p_1)}{ds}$ and velocity $u_s = \{(u_x - U)^2 + u_y^2\}^{1/2}$ in x and y-directions, it follows from formula (6f)

$$\frac{-h\frac{\partial}{\partial x}(p - p_1)}{\rho(u_x - U)\{(u_x - U)^2 + u_y^2\}^{1/2}} = n_0\left[\frac{\rho\{(u_x - U)^2 + u_y^2\}^{1/2}h}{\eta}\right]^{m_0} \qquad (7c)$$

$$\frac{-h\frac{\partial}{\partial y}(p - p_1)}{\rho u_y\{(u_x - U)^2 + u_y^2\}^{1/2}} = n_0\left[\frac{\rho\{(u_x - U)^2 + u_y^2\}^{1/2}h}{\eta}\right]^{m_0} \qquad (7d)$$

In the original limiting case of parallel flow directions (where the flow component u_y, at right angles to the sliding direction of the surface, reduces to zero), equation (7a) reduces to (6a) and equation (7c) to (6b).

Equations (7a–d) have been written in such a way that the fictitious pressure gradients can be easily eliminated. Therefore, equations can be derived giving gradients of the actual pressure in the lubricant film, as follows:

$$-\frac{h^2}{\eta U}\frac{\partial p}{\partial x}\left(\frac{\eta}{\rho U h}\right)^{1+m_0} = {}^1\!/_2 n_0\left\{U_x(U_x^2 + U_y^2)^{\frac{1+m_0}{2}} + (U_x - 1)((U_x - 1)^2 + U_y^2)^{\frac{1+m_0}{2}}\right\} \qquad (8a)$$

$$-\frac{h^2}{\eta U}\frac{\partial p}{\partial y}\left(\frac{\eta}{\rho U h}\right)^{1+m_0} = {}^1\!/_2 n_0\left\{U_y(U_x^2 + U_y^2)^{\frac{1+m_0}{2}} + U_y((U_x - 1)^2 + U_y^2)^{\frac{1+m_0}{2}}\right\} \qquad (8b)$$

where

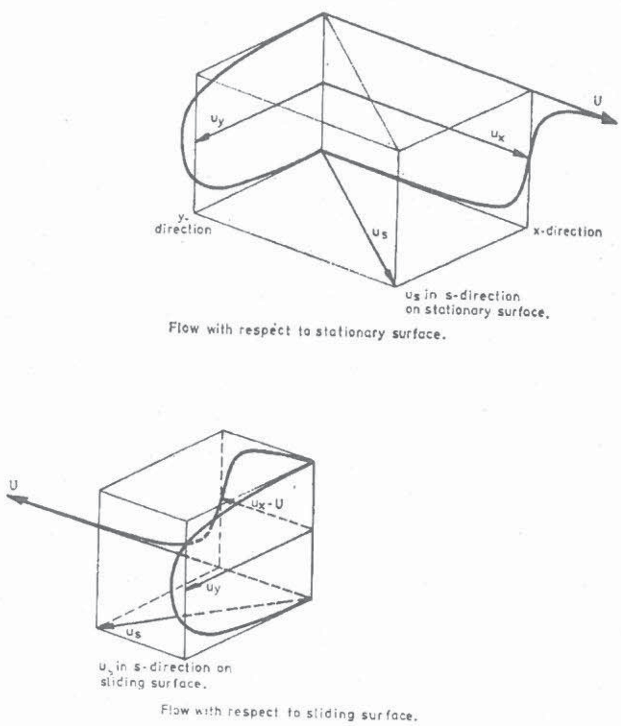

Fig. 4 Flow in mutually perpendicular directions

$U_x = \frac{u_x}{U}$ and $U_y = \frac{u_y}{U}$ are normalized velocities of flow. (8b)

This way of formulating the basic equations for the turbulent lubricant film has advantages in comparing theoretical and experimental results. It can be shown from the derivation that only a minimum of experimental information is required.

(a) It is possible to derive the magnitudes of n_0 and m_0 from a flow experiment in which pressure flow is the only type of flow to occur. It is particularly the magnitude of n_0 that matters. The magnitude of m_0 need not be known accurately, provided the absolute value is much smaller than unity, since in the formulas (8a) and (8b) m_0 appears only in the $^1/_2(1 + m_0)$th power.

(b) It is possible to derive the quantities n_1 and m_1 from an experiment with drag flow. If however, only the pressure build-up in a bearing is required, it would suffice to determine whether n_1 and m_1 do not deviate too much from n_0 and m_0, respectively. In fact, it proved possible, thanks to the introduction of a fictitious pressure, to derive the equations (8a) and (8b) in which the quantities n_1 and m_1 no longer appear.

Inertia Effects Other Than Turbulence

A disadvantage of the foregoing equations might be the fact that one of the two bearing surfaces has been assumed to be stationary. In many bearings either surface might move with respect to the lubricant film as a body and it is convenient to attach a new frame of reference **x, y** to that body of the film. Now, let the velocity of the new coordinate system **x, y**, be in the x-direction. Further, let the meaning of symbol U no longer be restricted to sliding speed as in the previous frame of reference but let the meaning of it be extended to the sum of the speeds of the two surfaces (u_1 and u_2) with respect to the new coordinate system **x, y**: $U = u_1 + u_2$ or $U_1 + U_2 = 1$ and $U_\mathbf{y} = U_y$, $U_\mathbf{x} = U_x + U_1$. This generalizing transformation yields:

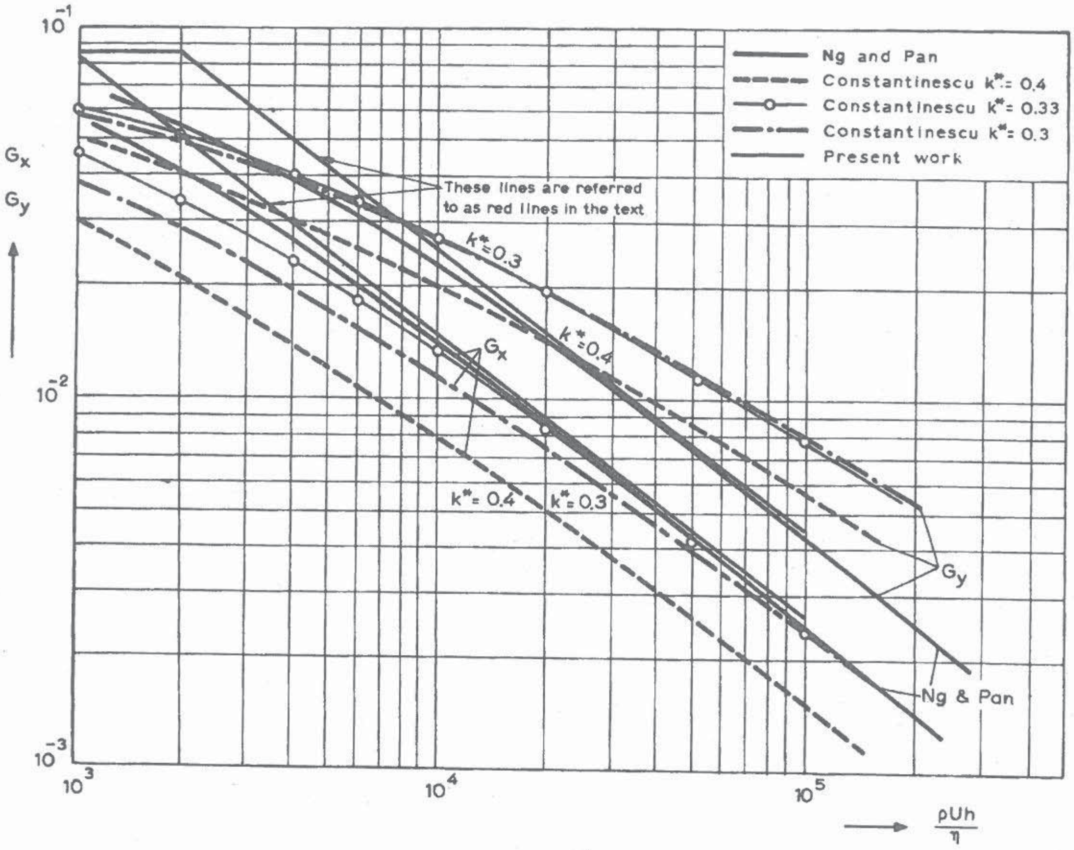

Fig. 5 Variation of G_x and G_y with $\frac{\rho U h}{\eta}$ if drag flow is dominant

$$-\frac{h^2}{\eta U}\frac{\partial p}{\partial x}\left(\frac{\eta}{\rho U h}\right)^{1+m_0} = {}^1/_2 n_0 \left\{(U_x - U_1)((U_x-U_1)^2+U_y^2)^{\frac{1+m_0}{2}} + (U_x-U_2)((U_x-U_2)^2+U_y^2)^{\frac{1+m_0}{2}}\right\} \quad (9a)$$

$$-\frac{h^2}{\eta U}\frac{\partial p}{\partial y}\left(\frac{\eta}{\rho U h}\right)^{1+m_0} = {}^1/_2 n_0 \left\{U_y((U_x - U_1)^2+U_y^2)^{\frac{1+m_0}{2}} + U_y((U_x-U_2)^2+U_y^2)^{\frac{1+m_0}{2}}\right\} \quad (9b)$$

and

$$U_1 + U_2 = 1 \quad (9c)$$

The presence of inertia effects other than those inherent in turbulence may be incorporated most concisely in equations (9a) and (9b) thanks to the facts that the pressure is explicit and that the coordinate system is stationary with respect to the shape of the wedge. To this end the following terms must be added to the right-hand side of equations (9a) and (9b)

$$+\left(\frac{\eta}{\rho U h_0}\right)^{m_0}\left\{\frac{h}{U}\frac{\partial U_x}{\partial t} + hU_x\frac{\partial U_x}{\partial x} + hU_y\frac{\partial U_x}{\partial y}\right\} \quad (10a)$$

and

$$+\left(\frac{\eta}{\rho U h_0}\right)^{m_0}\left\{\frac{h}{U}\frac{\partial U_y}{\partial t} + hU_x\frac{\partial U_y}{\partial x} + hU_y\frac{\partial U_y}{\partial y}\right\} \quad (10b)$$

respectively. In terms (10a) and (10b) the acceleration terms $\left(\frac{\partial U_x}{\partial t}, \text{ etc.}\right)$ are correct from a physical viewpoint. However, the convective inertia terms $\left(U_x\frac{\partial U_x}{\partial x}, \text{ etc.}\right)$ are not entirely correct. This is due to the fact that the products of averaged flow velocities U_x and U_y instead of averaged products of local flow velocities are inserted in (10a) and (10b). However, profiles of the local flow velocities are known to be rather blunt. Thus, the difference between the product of two averaged velocities and the averaged product of two local velocities can be expected to be small. Indeed, convective inertia terms are underestimated. However, such an error is permissible if the terms are a second order effect as far as pressure build-up is concerned.

Comparison of the Bulk-Flow Theory and Theories Based on the Law-of-Wall and the Mixing Length Concept

All equations, that in the various theories give pressure and flow in a turbulent lubricant film, are semiempirical and contain two empirical constants. The main difference between the bulk-flow theory and the other two theories, based on law-of-wall and mixing length concept, lies in the choice of the empirical constants. In the first theory, empirical constants are derived from measurements on bulk-flow between stationary or sliding plates and in pipes. In the other two theories empirical constants are derived from measurements of time-averaged flow velocity profiles between stationary or sliding plates and in pipes. In the first theory it is possible to derive the empirical constants without much insight in the mechanism of turbulent flow. In essence, the author's theory is not more than a generalization of the well-known dependence between friction-factor and Reynolds number first given by Blasius [2]. In the other two theories,

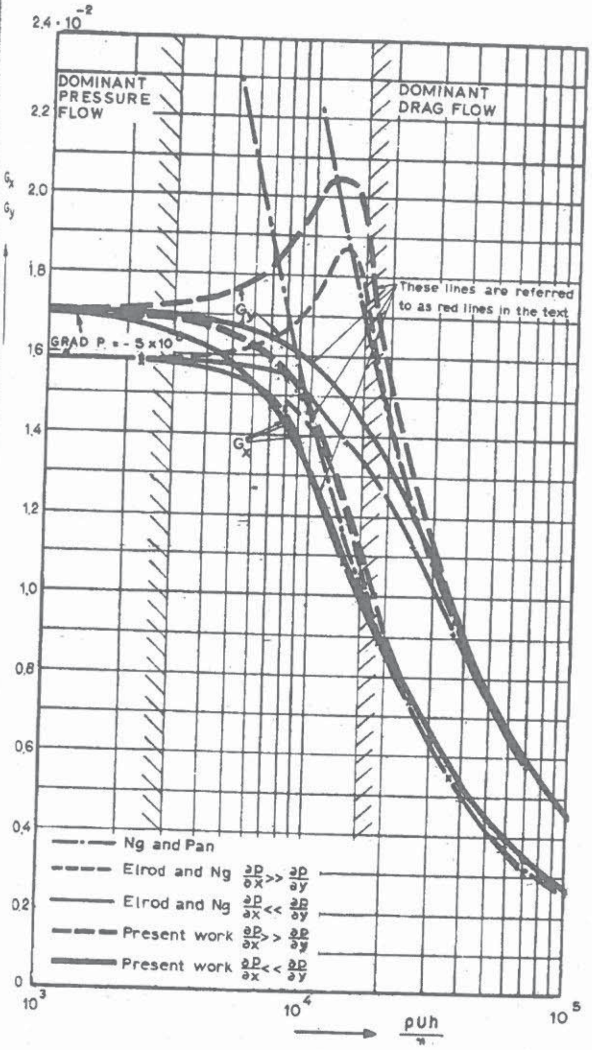

Fig. 6 Variations of G_x and G_y with $\frac{\rho U h}{\eta}$ at a great pressure gradient

empirical constants cannot but be derived after having introduced a model of the turbulence mechanism. Indeed, the introduction of the law-of-wall leads to a quantitative determination of the universal velocity profile near a wall and the introduction of the mixing length concept leads to the determination of the mixing length. However, any model accounting for the turbulence mechanism is only valid for certain limiting conditions. Thus the validity of these models is a matter of continuing dispute. It is outside the scope of this work to take part in this dispute. Therefore the practical results of the two theories based on the law-of-wall as worked out by Elrod and Ng[3] [8] and based on the mixing length concept as worked out by Constantinescu [5] are compared with the author's theory and it will not be discussed at length whether they have correctly applied realistic, physical models.

Practical results of the two theories have been presented by publishing graphs of magnitudes G_x en G_y derived from the following two equations:

$$U_x = -G_x \frac{h^2}{\eta U} \frac{\partial p}{\partial x} + {}^1/_2 \qquad (11a)$$

$$U_y = -G_y \frac{h^2}{\eta U} \frac{\partial p}{\partial y} \qquad (11b)$$

[3] An earlier, linearized, version of this theory is by Ng and Pan [10].

The first equation gives the dimensionless, mean flow velocity in the direction of sliding. It is seen to consist of a pressure or Poiseuille flow component and a drag or Couette flow component. The second equation (11*b*) gives the dimensionless, mean pressure flow velocity perpendicular to the direction of sliding. In the graphical results of the two theories it is shown that G_x and G_y reach a maximum value of $^1/_{12}$ for Reynolds numbers smaller than 1000 so that the equations become identical to the equations for laminar flow in that regime. In the turbulent regime equations for G_x and G_y can be derived by inserting U_x and U_y (equations (11*a*) and (11*b*)) into equations (8*a*) and (8*b*). The two equations take the following general form:

$$f_{1,2}\left(G_x, G_y, R, \frac{\partial P}{\partial X}, \frac{\partial P}{\partial Y}\right) = 0 \qquad (12a, b)$$

where

$$R = \frac{\rho U h}{\eta}$$

$$\frac{\partial P}{\partial X} = \frac{\rho h^3}{\eta^2} \frac{\partial p}{\partial x}$$

$$\frac{\partial P}{\partial Y} = \frac{\rho h^3}{\eta^2} \frac{\partial p}{\partial y}$$

From these equations, numerical results are derived and plotted in graphs originally used for theories based on law-of-wall and mixing length.

In the first place, it will be assumed that the pressure flow component is much smaller than the drag flow component:

$$G_x \frac{\partial P}{\partial X} \ll {}^1/_2 R$$

$$G_y \frac{\partial P}{\partial Y} \ll {}^1/_2 R$$

These conditions are characteristic of self-acting bearings operating at moderate eccentricities. Under these assumptions, equations (12*a*) and (12*b*) can be simplified to

$$G_x = \frac{2^{1+m_0}}{n_0(2 + m_0)} R^{-(1+m_0)} \qquad (13a)$$

$$G_y = \frac{2^{1+m_0}}{n_0} R^{-(1+m_0)} \qquad (13b)$$

in which $n_0 = 0.066$ and $m_0 = -0.25$ for smooth surfaces at Reynolds numbers smaller than 10^5, see Davies and White [7] and others. Graphical results are presented in Fig. 5.[4] Red lines are results from the bulk-flow theory and based on the above two formulas. Drawn black lines are results from the law-of-wall (Ng and Pan) and dotted black lines are based on the mixing length concept (Constantinescu). It can be seen that the dependency between G_x and $R = \frac{\rho U h}{\eta}$ is roughly identical for the three theories in a regime characterized by $10^4 < R < 10^5$. However, the mixing length theory appears to deviate from the other two theories for the dependency between G_y and R at different values of the mixing length constant k^x. Thus the mixing length theory appears to be less reliable than the other two theories.

Next, no limitations are placed upon the ratio between the pressure flow component and the drag flow component. In Figs. 6 and 7, values for G_x and G_y derived from the general equations (12*a*) and (12*b*) are plotted for several values of the dimensionless pressure gradient and with Reynolds number based on sliding

[4] Fig. 6.14 from Constantinescu [5].

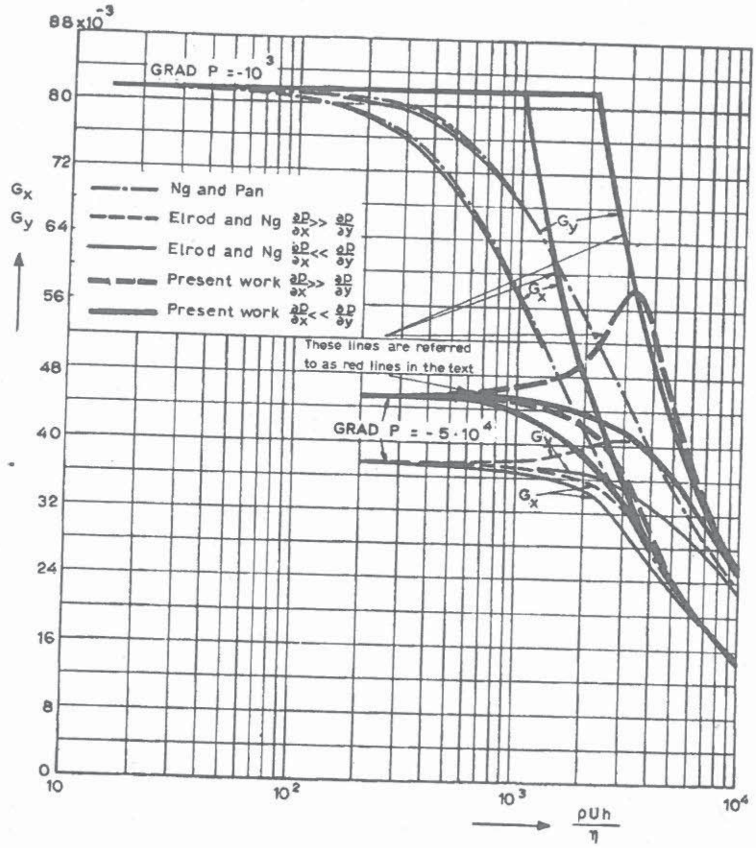

Fig. 7 Variation of G_x and G_y with $\frac{\rho U h}{\eta}$ at two small pressure gradients

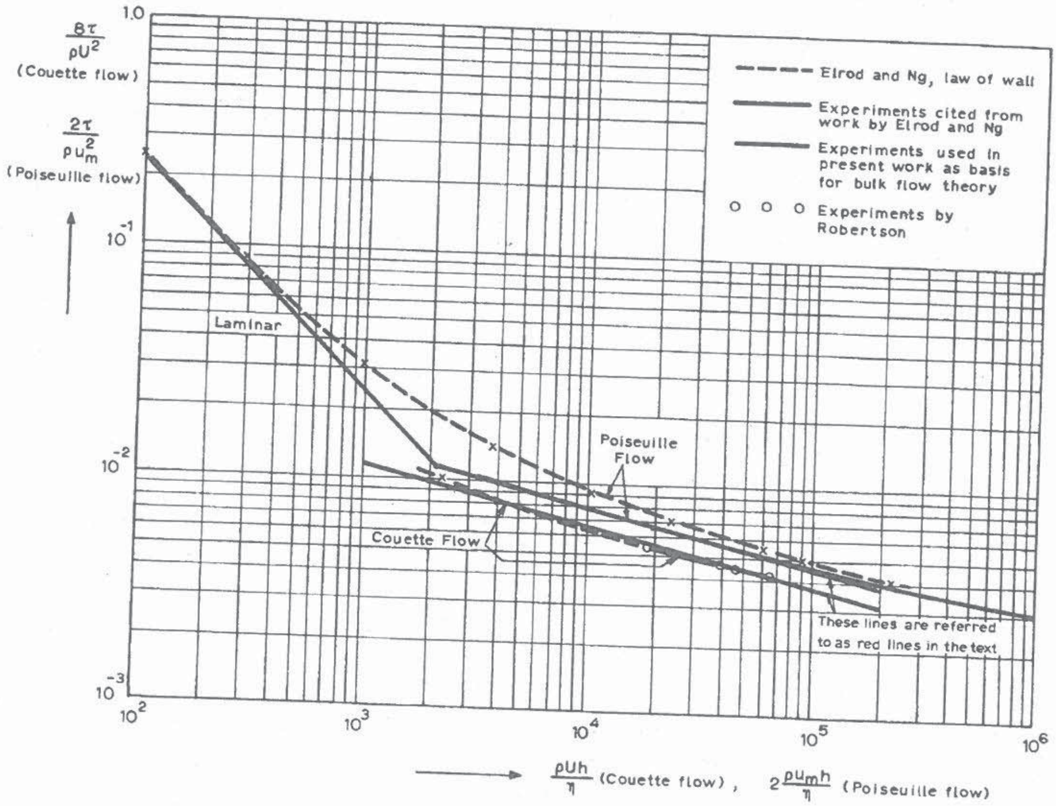

Fig. 8 Variations of friction factors with Reynolds numbers

speed as the independent variable. In these figures, red lines are results from the bulk-flow theory and black lines are results from the law-of-wall theory. Drawn lines represent cases in which pressure gradient and direction of sliding are roughly perpendicular or $\frac{\partial P}{\partial Y} \gg \frac{\partial P}{\partial X}$. Dotted lines represent cases in which pressure gradient and sliding speed are almost parallel or $\frac{\partial P}{\partial X} \gg \frac{\partial P}{\partial Y}$.

It can be deduced from the two figures that it is unimportant how the pressure gradient and sliding speed are directed with respect to each other for $R \rightarrow \infty$ and $R \rightarrow 0$, representing cases with a dominant flow due to a sliding surface and a dominant flow under the influence of a pressure gradient, respectively. Drawn and dotted lines deviate in a regime where the two flow components have the same order of magnitude. From a comparison between red lines representing bulk-flow theory and black lines representing law-of-wall theory, it follows that the two theories show the general behavior outlined in the foregoing and that they are in accordance in Fig. 6.

A comparison between red and black lines in Fig. 7 also shows that discrepancies are present for the smaller Reynolds numbers ($R < 10^4$) and for the smaller dimensionless pressure gradients $\left(\frac{\partial P}{\partial X} \text{ and } \frac{\partial P}{\partial Y} < 10^5\right)$.

Such increasing discrepancies for decreasing Reynolds numbers can also be seen in Fig. 5, which is representative of a dominant drag flow.

It remains to be explained why good accordance does not extend over the complete turbulent regime and it remains to be found which of the two theories is better in that part of the regime in which no good accordance could be found. The explanation can be found in Elrod and Ng's own work, see Fig. 8. In this figure black dotted lines are based on law-of-wall theory and black and red drawn lines are the result of experiments. The vertical coordinate gives the friction factor

$$\frac{2\tau}{\rho u_m^2} \quad \text{or} \quad \frac{8\tau}{\rho U^2} \quad \text{and}$$

the horizontal gives the Reynolds number

$$\frac{2\rho u_m h}{\eta} \quad \text{or} \quad \frac{\rho U h}{\eta}.$$

When discussing this figure Elrod and Ng state that the law-of-wall theory is asymptotically correct in the limits of high and low Reynolds numbers. It is not to be expected, of course, that any agreement be achieved in the transition region which extends up until $\frac{\rho U h}{\eta} = 10^4$.

Thus, it follows from Fig. 8 that a theory based on law-of-wall is not so good in accordance with experiments than the bulk-flow theory, which is directly based on experiments. The lack in good accordance is restricted to the smaller Reynolds numbers.

The general conclusion is that the bulk-flow theory is more reliable than the other two theories.

Conclusions

1 The bulk-flow theory is in excellent accordance with the law-of-wall theory for turbulent flow in bearing films at the greater Reynolds numbers.

2 The present theory is marginally better for turbulent flow at the smaller Reynolds numbers.

3 Equations for turbulent flow in bearing films have now been presented in closed form, (8a) and (8b).

4 No physical models for the turbulence mechanism have been used when developing the new theory.

5 Empirical constants (2) used in the theory can be derived from bulk-flow measurements and do not require the determination of flow velocity profiles.

6 The present theory is developed for turbulent flow in bearing films interposed between smooth surfaces. Unlike other theories, it can easily be extended to flow between rough and grooved surfaces.

Acknowledgments

The present work on turbulent lubricant films was carried out at the Institute TNO for Mechanical Constructions. It was sponsored by the Association Euratom—TNO/RCN and administered by A. H. de Haas van Dorsser. When developing the theory advice was given by Prof. H. Blok, University of Technology, Delft; R. A. Burton, then at U. S. Office of Naval Research, London; Prof. J. O. Hinze, University of Technology, Delft; and A. Verduin, Projectgroup Nuclear Energy TNO, The Hague. Results of this work have been used for designing bearings for Stork liquid sodium pumps.

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DISCUSSION

V. N. Constantinescu[5]

One of the main results of the turbulent theories based on the detailed analysis of the shear flow including turbulent stresses [3], [8], [10] is the fact that although the phenomenon is locally nonlinear, quasi-linear relations hold valid for some global characteristics such as mean velocities, flow versus pressure gradients, friction stresses versus mean velocities and versus pressure gradients, etc. Starting from this point, a bulk theory can indeed be developed, by using any theory which is able to produce the mentioned global relationships or, as done in this paper, by

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using experimental data taken from some other similar flows.

Figs. 6 and 7 point out that even when superposition of stresses due to Poiseuille and Couette flow exists, nonlinear behavior in the relationship flow versus pressure gradients can be emphasized for large pressure gradients. It may be shown that a similar behavior exists for friction stresses τ_a τ_b versus pressure gradients. Indeed [5] although in the absence of convective inertia forces the relation

$$\tau_b - \tau_a = h\frac{\partial p}{\partial x}$$

is always valid, linear dependence of both τ_a, τ_b on $\partial p/\partial x$ is valid approximately for the same conditions when parameters G_x, G_y are depending only on Couette Reynolds number. In other words, it may be shown [5] that in dimensionless form

$$\bar{\tau}_{b,a} = \bar{\tau}_c \pm B_x; \; \bar{\tau} = \frac{h\tau}{\eta U}; \; B_x = -\frac{h^2}{\eta U}\frac{\partial p}{\partial x}$$

if (B_x) does not assume too large values. The Couette friction stress $\bar{\tau}_c$ is a function of the Couette Reynolds number, e.g., when using the mixing length approach

$$-\bar{\tau}_c = 1 + 0.012\left(\frac{\rho U h}{\eta}\right)^{0.94}$$

It would be of interest if the author plot $\bar{\tau}_a$ and $\bar{\tau}_b$ as functions of parameter B_x for various Reynolds numbers and compare his results with those given, for example, in reference [5].

Author's Closure

The author wishes to thank Dr. Constantinescu for his discussion.

The equation based on the mixing length for the Couette friction stress τ_c is not in agreement with the one based on experiments, cf. equation (3) in the paper using $n_1 = 0.055$ and $m_0 = -0.25$.

A comparison between wall shear stresses τ_a and τ_b based on the bulk flow theory and the mixing length approach is of great interest. However, the author does not have available reference [5] at the moment of writing his reply. Therefore, the comparison will be dealt with in a future letter to the editor of JOURNAL OF LUBRICATION TECHNOLOGY.