Dynamics COE2004

Three-Dimensional Kinetics of a Rigid Body

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2022 / 11 / 29

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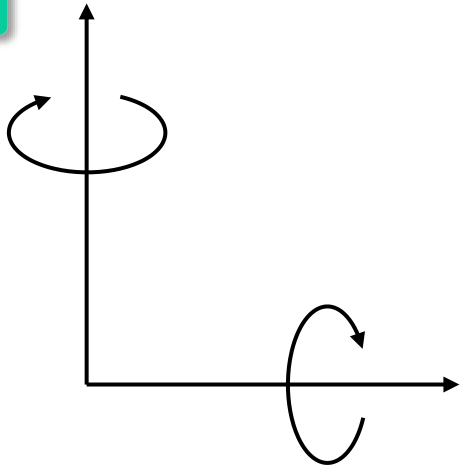
I Moments and Products of Inertia

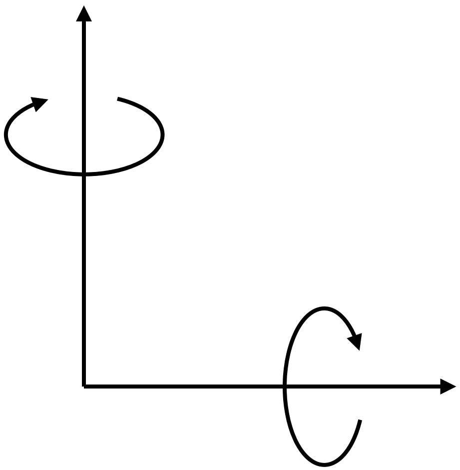
Moment of Inertia

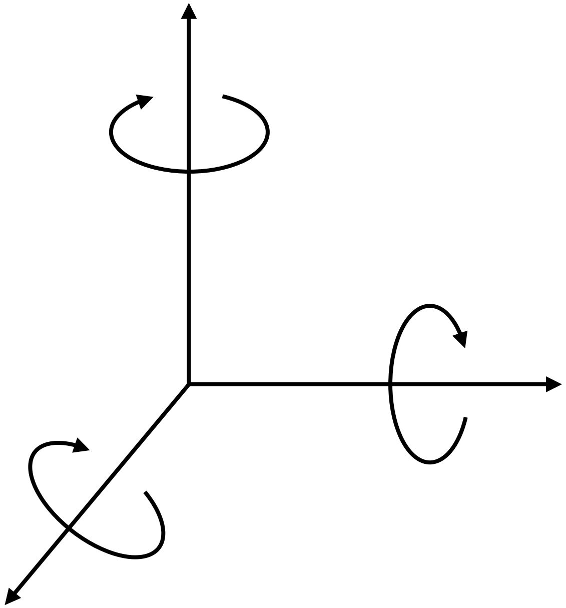
Planar motions in two-dimensional space



- Two directions of basis
- Two perpendicular (Rotational) directions with respect to basis
- Four equations

Moment of Inertia

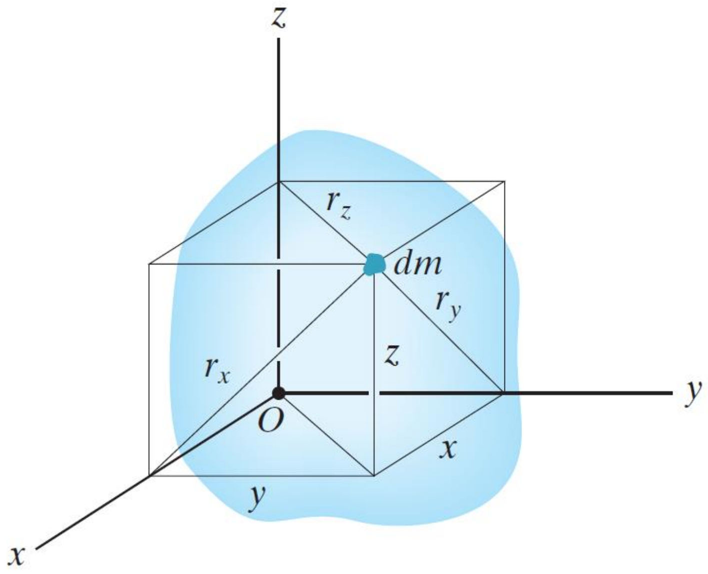




- Two directions of basis
- Two perpendicular (Rotational) directions with respect to basis
- Four equations

- Three directions of basis
- Three perpendicular (Rotational) directions with respect to basis
- Six equations

Moment of Inertia



$$r_x = \sqrt{y^2 + z^2}$$

$$r_y = \sqrt{x^2 + z^2}$$

$$r_z = \sqrt{x^2 + y^2}$$

Mass moment of inertia of element, dm

- x-axis

$$dI_{xx} = r_x^2 dm = (y^2 + z^2)dm$$

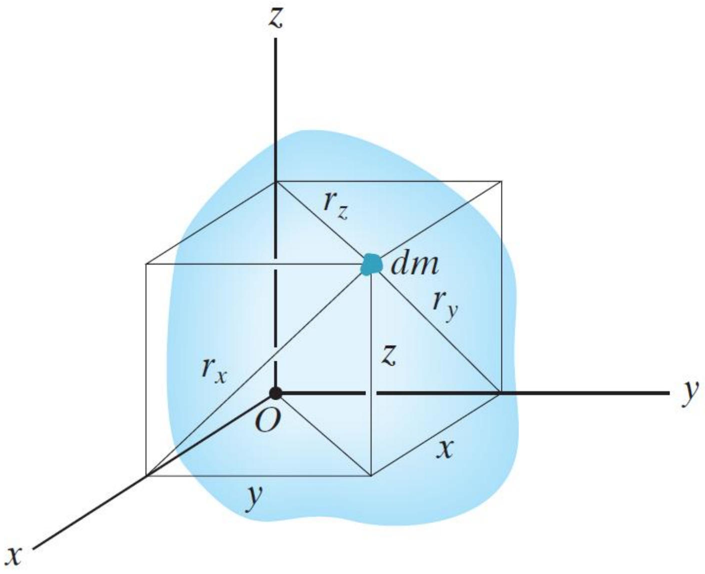
- y-axis

$$dI_{yy} = r_y^2 dm = (x^2 + z^2)dm$$

- z-axis

$$dI_{zz} = r_z^2 dm = (x^2 + y^2)dm$$

Moment of Inertia



Mass moment of inertia of element, dm

- x-axis

$$dI_{xx} = r_x^2 dm = (y^2 + z^2)dm$$

- y-axis

$$dI_{yy} = r_y^2 dm = (x^2 + z^2)dm$$

- z-axis

$$dI_{zz} = r_z^2 dm = (x^2 + y^2)dm$$

Moment of inertia for the rigid body

Positive quantity

- x-axis

$$I_{xx} = \int_m r_x^2 dm = \int_m (y^2 + z^2)dm$$

- y-axis

$$I_{yy} = \int_m r_y^2 dm = \int_m (x + z^2)dm$$

- z-axis

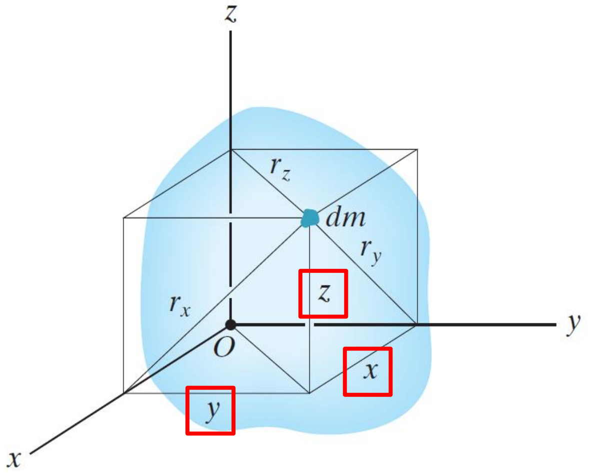
$$I_{zz} = \int_m r_z^2 dm = \int_m (x^2 + y^2)dm$$

Product of Inertia

Product of inertia

- The product of inertia for a differential element dm with respect to a set of two orthogonal planes
  : The product of the mass of the element and the perpendicular distances from the planes

- ex) x about y-z plane / y about x-z plane



Product of Inertia

Product of inertia

- The product of inertia for a differential element dm with respect to a set of two orthogonal planes
  : The product of the mass of the element and the perpendicular distances from the planes

$$dI_{xy} = xydm$$
$$dI_{xy} = dI_{yx}$$

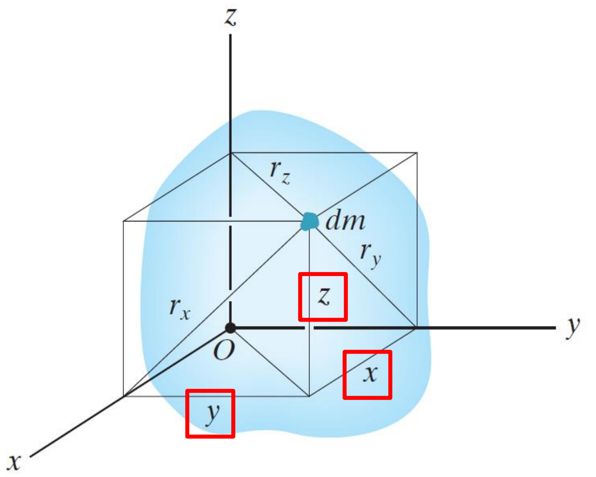
$$dI_{yz} = yzdm$$
$$dI_{yz} = dI_{zy}$$

$$dI_{xz} = xzdm$$
$$dI_{xz} = dI_{zx}$$

$$I_{xy} = I_{yx} = \int_m xydm$$

$$I_{yz} = I_{zy} = \int_m yzdm$$

$$I_{xz} = I_{zx} = \int_m xzdm$$



Product of Inertia

Moment of inertia

$$I_{xx} = \int_m r_x^2 dm = \int_m (y^2 + z^2) dm$$

$$I_{yy} = \int_m r_y^2 dm = \int_m (x + z^2) dm$$

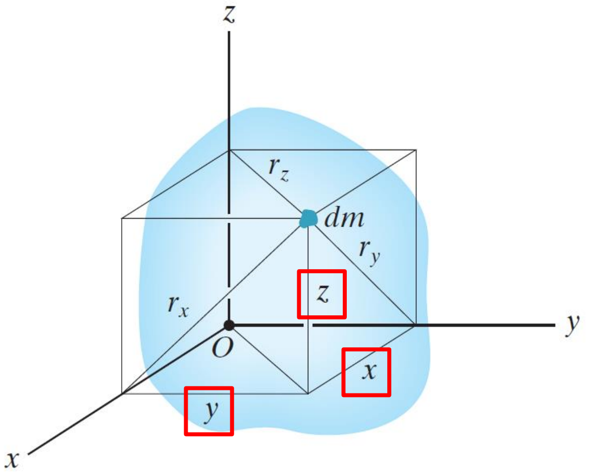
$$I_{zz} = \int_m r_z^2 dm = \int_m (x^2 + y^2) dm$$

Product of inertia

$$I_{xy} = I_{yx} = \int_m xydm$$

$$I_{yz} = I_{zy} = \int_m yzdm$$

$$I_{xz} = I_{zx} = \int_m xzdm$$



Product of Inertia

Angular momentum in a rigid body with (p, q, r) coordinate

$$H_p = I_{pp}\omega_p + I_{pq}\omega_q + I_{pr}\omega_r$$

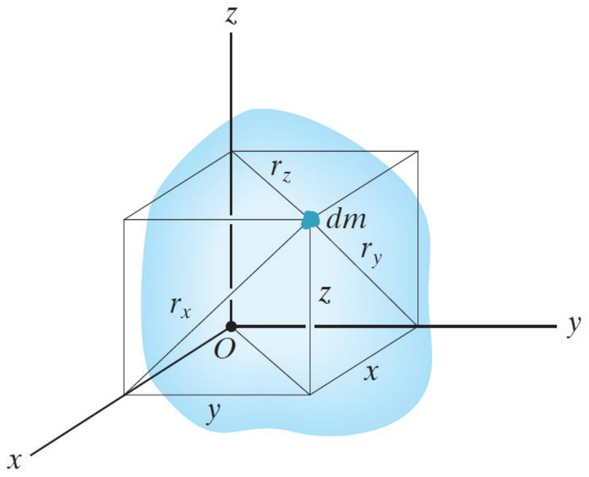
$\cdots$ ***Angular momentum about p axis***

$$H_q = I_{qp}\omega_p + I_{qq}\omega_q + I_{qr}\omega_r$$

$\cdots$ ***Angular momentum about q axis***

$$H_r = I_{rp}\omega_p + I_{rq}\omega_q + I_{rr}\omega_r$$

$\cdots$ ***Angular momentum about r axis***



Product of Inertia

$$I_{xx} = \int (y^2 + z^2)dm \qquad I_{xy} = \int xydm \qquad I_{xz} = \int xzdm$$

$$I_{yx} = \int yxdm \qquad I_{yy} = \int (x^2 + z^2)dm \qquad I_{yz} = \int yzdm$$

$$I_{zx} = \int zxdm \qquad I_{zy} = \int zydm \qquad I_{zz} = \int (x^2 + y^2)dm$$

Moment of inertia about a plane

- Physical properties which is resistant to rotational motions about an axis

Product of inertia

- Physical properties which indicates the effect of rotational motions about one axis on another axis

Product of Inertia

$$I_{xx} = \int (y^2 + z^2) dm \qquad I_{xy} = \int xydm \qquad I_{xz} = \int xzdm$$

$$I_{yx} = \int yxdm \qquad I_{yy} = \int (x^2 + z^2) dm \qquad I_{yz} = \int yzdm$$

$$I_{zx} = \int zxdm \qquad I_{zy} = \int zydm \qquad I_{zz} = \int (x^2 + y^2) dm$$

Moment of inertia about a plane

- Physical properties which is resistant to rotational motions about an axis

Product of inertia

- Physical properties which indicates the effect of rotational motions about one axis on another axis

Inertia Tensor

With 3x3 matrix

$$[H] = [I][\omega]$$

$$[H] = \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix}, [\omega] = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

$$[I] = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}$$

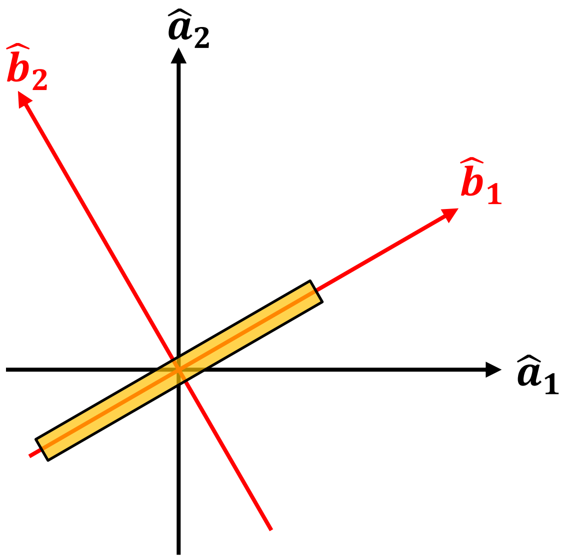
Inertia Tensor

Inertia tensor

- The inertia tensor has a unique set of values for a body when it is determined for each location of the origin *O* and orientation of the coordinate axes

Inertia Tensor

+α) If non-diagonal components are zero?



New directions

- New directions (Principal directions) $(\hat{b}_1, \hat{b}_2, \hat{b}_3)$

$$[I^*] = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

$$[H] = \begin{bmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{bmatrix}$$

$$\vec{H}^G = I_1\vec{\omega}_1\hat{b}_1 + I_2\vec{\omega}_2\hat{b}_2 + I_3\vec{\omega}_3\hat{b}_3$$

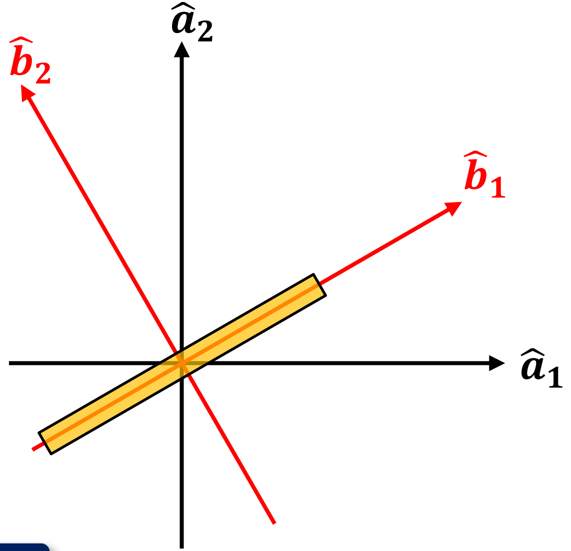
with $\vec{\omega} = \vec{\omega}_1\hat{b}_1 + \vec{\omega}_2\hat{b}_2 + \vec{\omega}_3\hat{b}_3$

Inertia Tensor

Differential of angular momentum vector

$$ {}^{A}\frac{d}{dt}\left(\vec{H}^{G}\right) = {}^{B}\frac{d}{dt}\left(\vec{H}^{G}\right) + \vec{\omega} \times \vec{H}^{G} $$

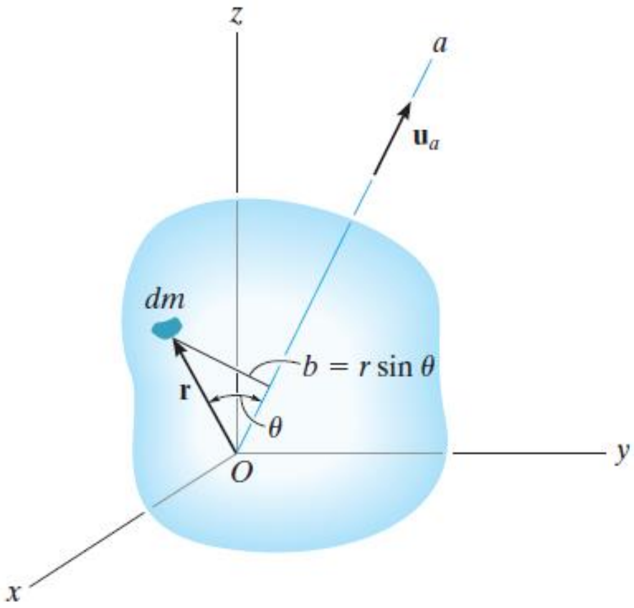
$$ \begin{cases} I_1\vec{\omega}_1 - (I_2 - I_3)\vec{\omega}_2\vec{\omega}_3 = M_1 \\ I_2\vec{\omega}_2 - (I_3 - I_1)\vec{\omega}_3\vec{\omega}_1 = M_2 \quad \cdots \textit{Euler equation} \\ I_3\vec{\omega}_3 - (I_1 - I_2)\vec{\omega}_1\vec{\omega}_2 = M_3 \end{cases} $$



Gyroscopic effect not on a plane

- Product of two angular velocities (nonlinear)
- When a moment is applied in the direction of rotation in another direction, angular velocity is generated in the other direction.

Moment of inertia about an arbitrary axis



- With 9 elements of the inertia tensor about x, y, z axes having an origin at *O*

Determine moment of inertia about O_a axis (unit vector: $\vec{u}_a$)

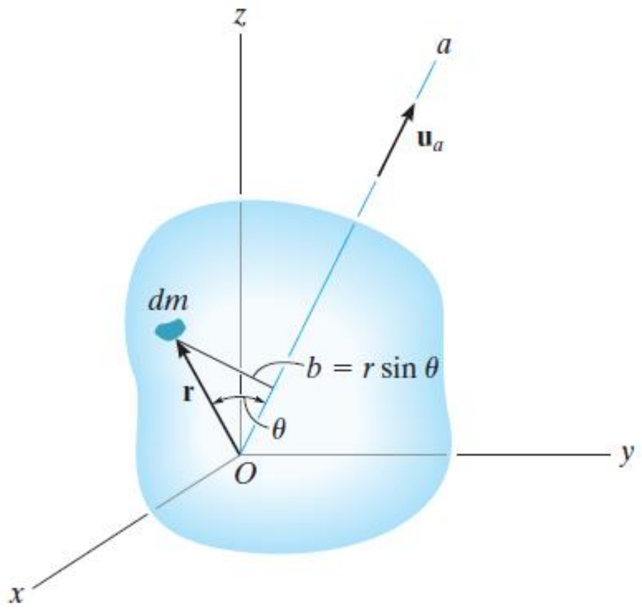
$$I_{Oa} = \int_m b^2\, dm$$
$$= \int_m |\vec{u}_a \times \vec{r}|^2\, dm$$
$$= \int_m |\vec{u}_a \times \vec{r}| \cdot |\vec{u}_a \times \vec{r}|\, dm$$

$$\vec{u}_a = u_x\hat{\imath} + u_y\hat{\jmath} + u_z\hat{k}$$
$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

Moment of inertia about an arbitrary axis

$$I_{Oa} = \int_m b^2\, dm$$
$$= \int_m |\vec{u}_a \times \vec{r}|^2\, dm$$
$$= \int_m |\vec{u}_a \times \vec{r}| \cdot |\vec{u}_a \times \vec{r}|\, dm$$

$$\vec{u}_a \times \vec{r} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_x & u_y & u_z \\ x & y & z \end{vmatrix}$$
$$= +(u_y z - u_z y)\hat{\imath} + (u_z x - u_x z)\hat{\jmath} + (u_x y - u_y x)\hat{k}$$



Moment of inertia about an arbitrary axis

$$I_{Oa} = \int_m b^2\, dm$$
$$= \int_m |\vec{u}_a \times \vec{r}|^2\, dm$$
$$= \int_m |\vec{u}_a \times \vec{r}| \cdot |\vec{u}_a \times \vec{r}|\, dm$$

$$\vec{u}_a \times \vec{r} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_x & u_y & u_z \\ x & y & z \end{vmatrix}$$
$$= +(u_y z - u_z y)\hat{\imath} +(u_z x - u_x z)\hat{\jmath} +(u_x y - u_y x)\hat{k}$$

$$I_{Oa} = \int_m \left[(u_y z - u_z y)^2 + (u_z x - u_x z)^2 + (u_x y - u_y x)^2\right] dm$$

$$= u_x^2 \int_m (y^2 + z^2)^2 dm + u_y^2 \int_m (z^2 + x^2)^2 dm + u_z^2 \int_m (x^2 + y^2)^2 dm$$
$$- 2u_x u_y \int_m xydm - 2u_y u_z \int_m yzdm - 2u_z u_x \int_m zxdm$$

Moment of inertia about an arbitrary axis

$$I_{Oa} = u_x^2 \int_m (y^2+z^2)^2 dm + u_y^2 \int_m (z^2+x^2)^2 dm + u_z^2 \int_m (x^2+y^2)^2 dm - 2u_x u_y \int_m xydm - 2u_y u_z \int_m yzdm - 2u_z u_x \int_m zxdm$$

With moment of inertia

$$I_{xx} = \int_m (y^2+z^2)^2 dm$$

$$I_{yy} = \int_m (x^2+z^2)^2 dm$$

$$I_{zz} = \int_m (x^2+y^2)^2 dm$$

And product of inertia

$$I_{xy} = I_{yx} = \int_m xydm$$

$$I_{yz} = I_{zy} = \int_m yzdm$$

$$I_{xz} = I_{zx} = \int_m xzdm$$

$$I_{Oa} = I_{xx}u_x^2 + I_{yy}u_y^2 + I_{zz}u_z^2 - 2I_{xy}u_x u_y - 2I_{yz}u_y u_z - 2I_{zx}u_z u_x$$

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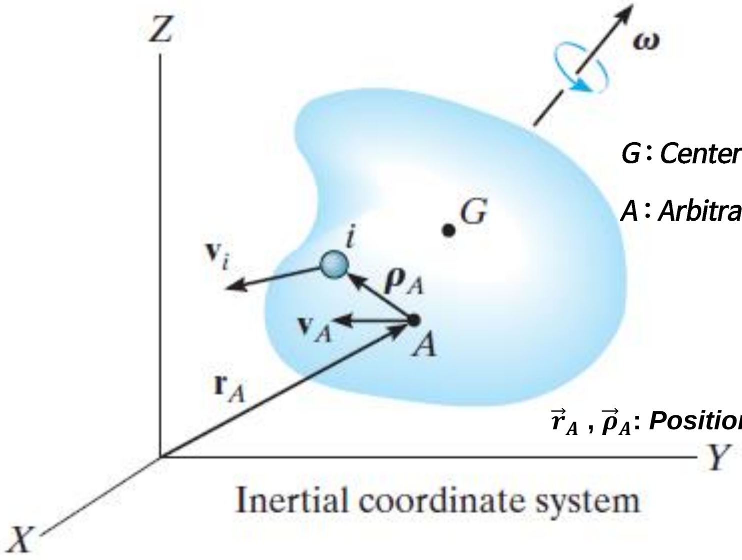
II Angular Momentum

Moment of inertia about an arbitrary axis



G: *Center of mass*

A: *Arbitrary point*

$\vec{r}_A$, $\vec{\rho}_A$: ***Position vectors***

Rigid body in three-dimensional space

- The rigid body which has a mass ***m*** and center of mass at ***G***
- X, Y, Z coordinate system represents an inertial frame
  : Axes are fixed or translate with a constant velocity
- The position vectors $\vec{r}_A$ and $\vec{\rho}_A$ are drawn from the origin of coordinates to point ***A*** and from ***A*** to the ***i*** th particle of the body

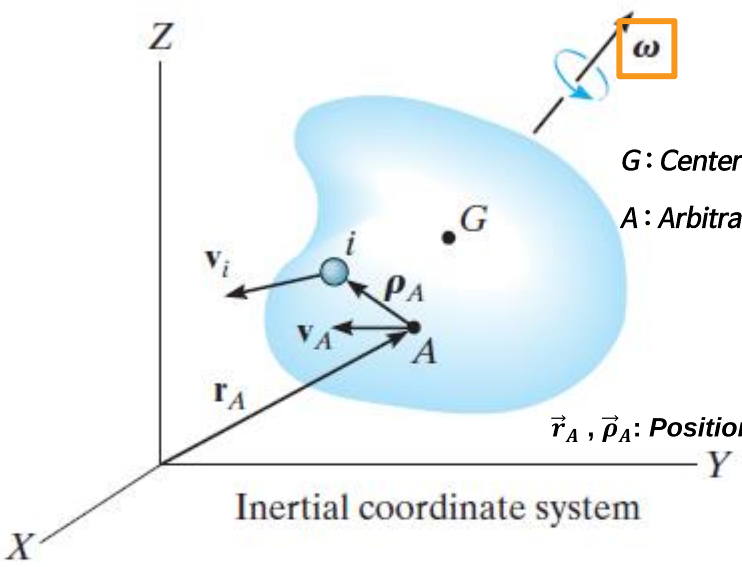
If) Particles mass: m_i

The angular momentum about point ***A*** is expressed as

$$\left(\overrightarrow{H_A}\right)_i = \vec{\rho}_A \times m_i \vec{v}_i$$

$\vec{v}_i$: *Velocity of particle measured from X, Y, Z coordinate system*

Moment of inertia about an arbitrary axis



G : *Center of mass*

A : *Arbitrary point*

$\vec{r}_A$, $\vec{\rho}_A$: ***Position vectors***

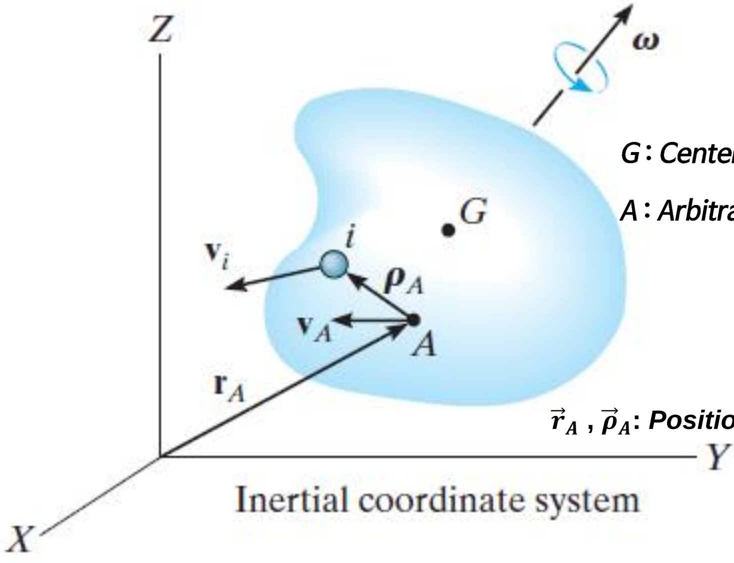
The angular momentum about point *A* is expressed as

$$\left(\overrightarrow{H_A}\right)_i = \vec{\rho}_A \times m_i \vec{v}_i$$

If the body has an angular velocity $\vec{\omega}$

$$\vec{v}_i = \vec{v}_A + \vec{\omega} \times \vec{\rho}_A$$

Moment of inertia about an arbitrary axis



G: *Center of mass*

A: *Arbitrary point*

$\vec{r}_A$, $\vec{\rho}_A$: ***Position vectors***

With

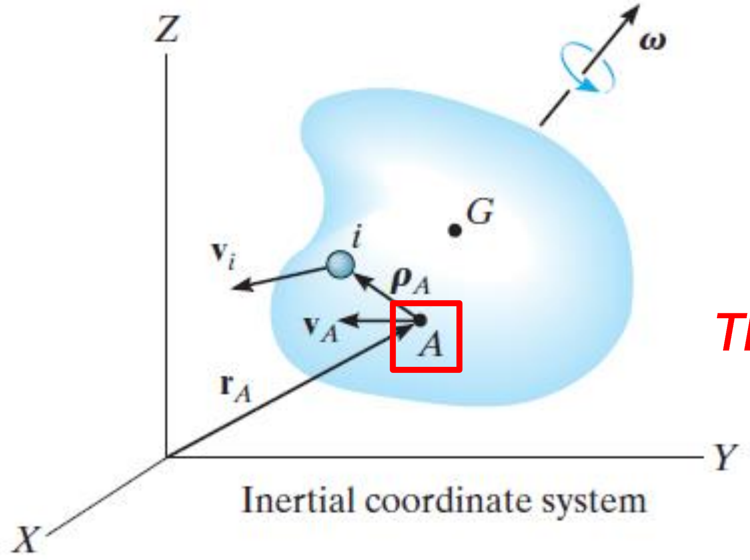
$(\vec{H}_A)_i = \vec{\rho}_A \times m_i \vec{v}_i$ and $\vec{v}_i = \vec{v}_A + \vec{\omega} \times \vec{\rho}_A$

$$
\begin{aligned}
(\vec{H}_A)_i &= \vec{\rho}_A \times m_i \vec{v}_i \\
&= \vec{\rho}_A \times m_i (\vec{v}_A + \vec{\omega} \times \vec{\rho}_A) \\
&= (\vec{\rho}_A m_i) \times \vec{v}_i + \vec{\rho}_A \times (\vec{\omega} \times \vec{\rho}_A) m_i
\end{aligned}
$$

Considering all particles of the body ($m_i \to dm$)

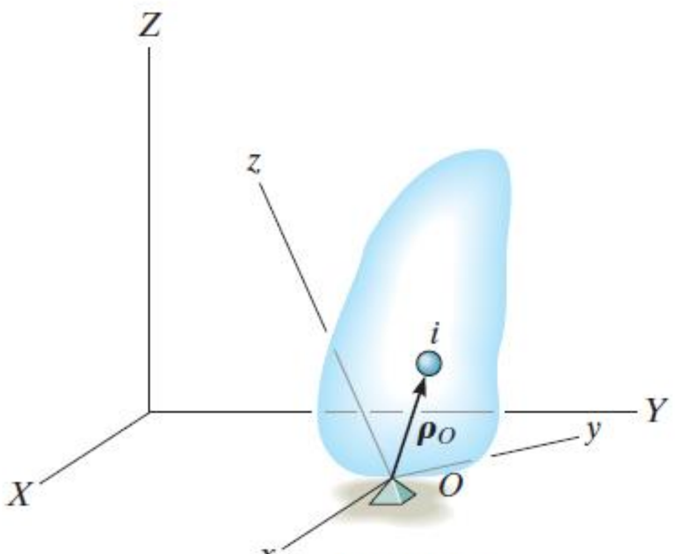
$$
\vec{H}_A = \left(\int_m \vec{\rho}_A dm \right) \times \vec{v}_A + \int_m \vec{\rho}_A \times (\vec{\omega} \times \vec{\rho}_A) dm
$$

Moment of inertia about an arbitrary axis



Then, point of A? ?

A at fixed point *O*



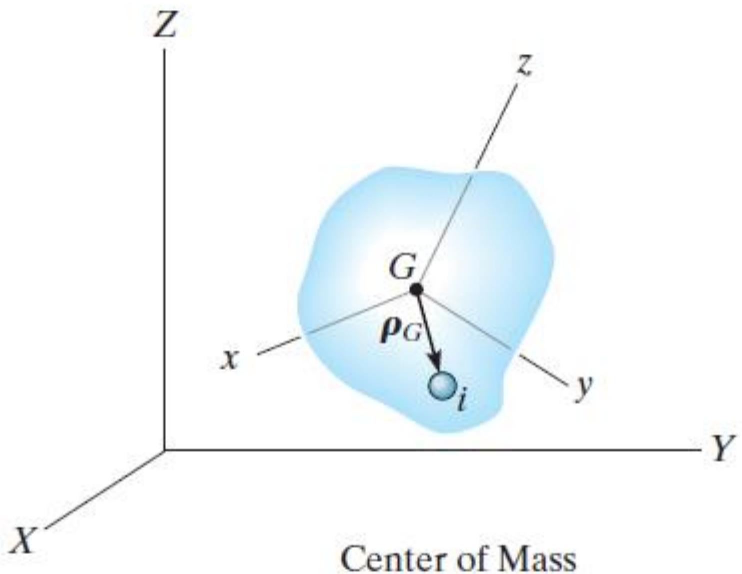
An arbitrary point *A* at fixed point

- An arbitrary point ***A*** is fixed at point ***O***
- If ***A*** becomes a fixed-point ***O*** in the body
  : $\vec{v}_A = 0$

$$\vec{H}_A = \left(\int_m \vec{\rho}_A dm\right) \times \vec{v}_A + \int_m \vec{\rho}_A \times (\vec{\omega} \times \vec{\rho}_A) dm$$

$$\vec{H}_O = \int_m \vec{\rho}_O \times (\vec{\omega} \times \vec{\rho}_O) dm$$

A at center of mass *G*



Center of Mass

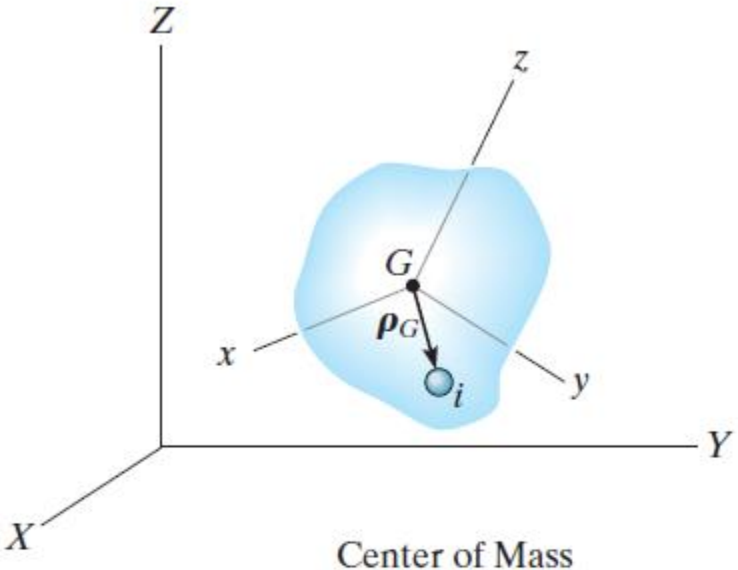
An arbitrary point *A* at center of mass

- An arbitrary point ***A*** is fixed at center of mass ***G***
- If ***A*** becomes a fixed-point at center of mass ***G*** in the body : $\int_m \vec{\rho}_A dm = 0$

$$\vec{H}_A = \left(\int_m \vec{\rho}_A dm\right) \times \vec{v}_A + \int_m \vec{\rho}_A \times (\vec{\omega} \times \vec{\rho}_A) dm$$

$$\vec{H}_G = \int_m \vec{\rho}_G \times (\vec{\omega} \times \vec{\rho}_G) dm$$

A at arbitrary point A



Center of Mass

An arbitrary point A

- In general, *A* can be a point other than *O* or *G*

$$\vec{H}_A = \left(\int_m \vec{\rho}_A dm\right) \times \vec{v}_A + \int_m \vec{\rho}_A \times (\vec{\omega} \times \vec{\rho}_A) dm$$

$$\vec{H}_A = \vec{\rho}_{G/A} \times m\vec{v}_G + \vec{H}_G$$

In practical aspect

Rectangular components of *H*

- The angular momentum should be expressed in terms of its scalar components
- Choose a second set of x, y, z axes having an arbitrary orientation relative to the X, Y, Z axes for general formulation about inertial frame

Expressing $\vec{H}$, $\vec{\rho}$, and $\vec{\omega}$ in terms of x, y, z components

$$H_x\hat{\imath} + H_y\hat{\jmath} + H_z\hat{k} = \int_m \left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right) \times \left[\left(\omega_x\hat{\imath} + \omega_y\hat{\jmath} + \omega_z\hat{k}\right) \times \left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right)\right] dm$$

In practical aspect

Expressing $\vec{H}$, $\vec{\rho}$, and $\vec{\omega}$ in terms of x, y, z components

$$H_x\hat{\imath} + H_y\hat{\jmath} + H_z\hat{k} = \int_m \left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right) \times \left[\left(\omega_x\hat{\imath} + \omega_y\hat{\jmath} + \omega_z\hat{k}\right) \times \left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right)\right] dm$$

$$\begin{aligned} H_x\hat{\imath} + H_y\hat{\jmath} + H_z\hat{k} = &\left[\omega_x \int_m (y^2 + z^2)dm - \omega_y \int_m xydm - \omega_z \int_m xzdm\right]\hat{\imath} \\ &+ \left[-\omega_x \int_m xydm + \omega_y \int_m (x^2 + z^2)dm - \omega_z \int_m yzdm\right]\hat{\jmath} \\ &+ \left[-\omega_x \int_m zxdm - \omega_y \int_m yzdm + \omega_z \int_m (x^2 + y^2)dm\right]\hat{k} \end{aligned}$$

In practical aspect

Equating the respective $\hat{\imath}, \hat{\jmath}, \hat{k}$ components and recognizing that the integrals represent the moments and products of inertia

$$H_x = I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z$$
$$H_y = -I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z$$
$$H_z = -I_{zx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z$$

- These equations can be simplified further if the x, y, z coordinate axes are oriented such that they become *principal axes of inertia* for the body at the point
- About *principal axes of inertia,* $I_{xy} = I_{yz} = I_{zx} = 0$
- If *principal moments of inertia,* $I_x = I_{xx}, I_y = I_{yy}, I_z = I_{zz}$ the three components of angular momentum become

$$H_x = I_x\omega_x, H_y = I_y\omega_y, H_z = I_z\omega_z$$


감사합니다