Dynamics COE2004

Gyroscope II

: What is Gyroscopic Motions?

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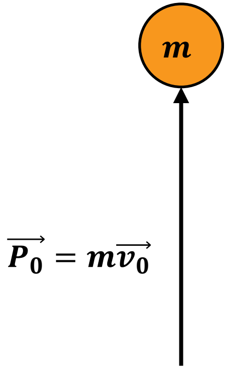
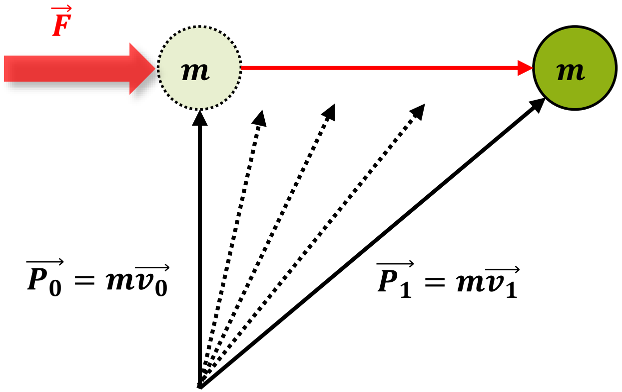
I Angular Momentum

Angular Momentum

A moving ball ($f = 0$)

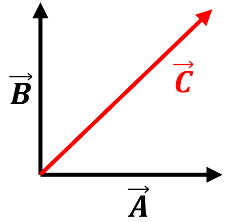


A forced ball ($f \neq 0$)



Force [kg·m/s²]
: Momentum [kg·m/s] change about unit time [s]

Vector summations



$$\vec{A} + \vec{B} = \vec{C}$$

$$\vec{P_0} + \vec{F} = \vec{P_1}$$

$$\Rightarrow \vec{F} = \vec{P_1} - \vec{P_0} = m\vec{v_1} - m\vec{v_0}$$

Derivative about time

$$\Delta\vec{P} = \Delta m\vec{v_n}$$

$$= \frac{d}{dt}(m\vec{v_n})$$

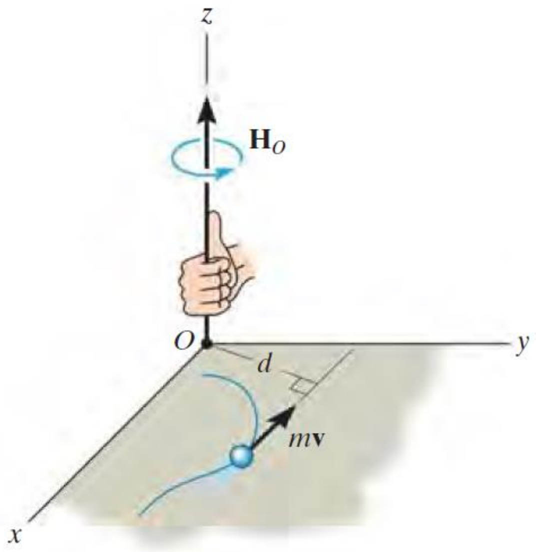
$$= \vec{v_n}\frac{dm}{dt} + m\frac{d\vec{v_n}}{dt}$$

$$\Rightarrow \Delta\vec{P} = m\vec{a}$$

$$\vec{F} = \frac{d\vec{P}}{dt}$$

Angular Momentum

Angular momentum



Scalar formulation

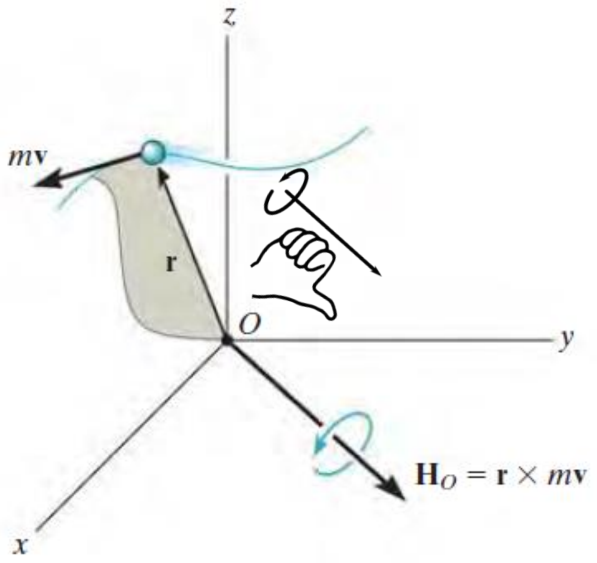
- If a particle moves along a curve lying in the x-y plane, the angular momentum at any instant can be determined about point *O*.
- The *magnitude* of $\mathbf{H}_O$ is

$$(\mathbf{H}_O)_z = (d)\,(mv)$$

- *d* is the moment arm or perpendicular distance from *O* to the line of action of $m\mathbf{v}$.
- The direction of $\mathbf{H}_O$ is defined by the right-hand rule.

Angular Momentum

Angular momentum



Vector formulation

- If the particle moves along a space curve, the cross product (vector product) can be used to determined the *angular momentum* about *O*.
- The *angular momentum* $\mathbf{H}_O$ is

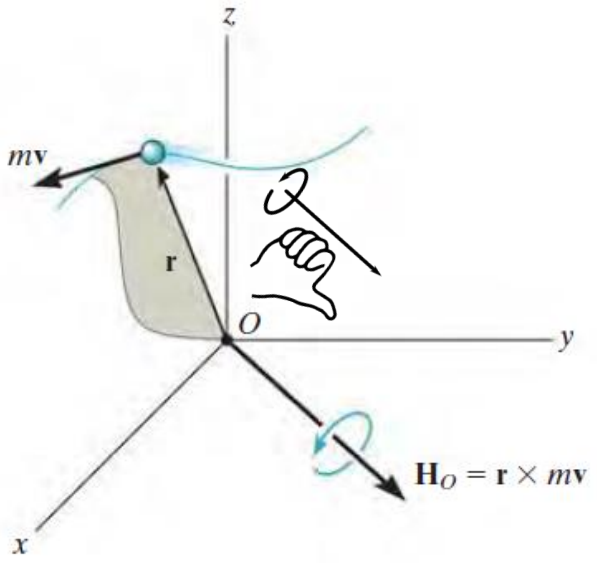
$$\mathbf{H}_O = \boldsymbol{r} \times m\boldsymbol{v}$$

- $\boldsymbol{r}$ denotes a position vector drawn from point O to the particle. H_O is perpendicular to the shaded plane containing $\boldsymbol{r}$ and $m\boldsymbol{v}$.

$$H_O = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ r_x & r_y & r_z \\ mv_x & mv_y & mv_z \end{vmatrix}$$

Angular Momentum

Angular momentum



Relation with Moment

- The moments of the forces about point *O* can be obtained by performing a cross product multiplication of each side of this equation by the position vector ***r***

$$\sum \boldsymbol{M_O} = \boldsymbol{r} \times \sum \boldsymbol{F} = \boldsymbol{r} \times m\dot{\boldsymbol{v}}$$

- With the angular momentum, $\mathbf{H}_O = \boldsymbol{r} \times m\boldsymbol{v}$

$$\dot{\boldsymbol{H}}_O = \frac{d}{dt}(\boldsymbol{r} \times m\boldsymbol{v}) = \dot{\boldsymbol{r}} \times m\boldsymbol{v} + \boldsymbol{r} \times m\dot{\boldsymbol{v}}$$

$$\sum \boldsymbol{M_O} = \dot{\boldsymbol{H}}_O$$

- The resultant force acting on the particle is equal to the time rate of change of the particle's linear momentum
- The resultant moment about point O of all forces acting on the particle is equal to the time rate of change of the particle's angular momentum

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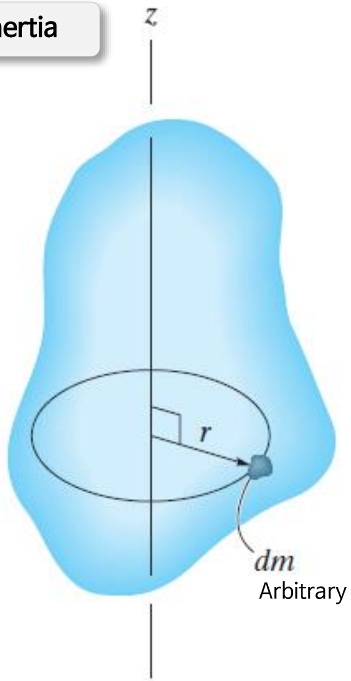
II Moment of Inertia

Moment of Inertia

Moment of Inertia



Arbitrary element

Relation with Moment

- In the rotational aspects, caused by a moment *M*, are governed by an equation of the form $\boldsymbol{M} = I\boldsymbol{\alpha}$
- $\boldsymbol{I}$ is termed the mass moment of inertia which is a measure of the resistance of a body to angular acceleration $(\boldsymbol{M} = I\boldsymbol{\alpha})$.
- Same way that mass is a measure of the body's resistance to acceleration $(\boldsymbol{F} = m\boldsymbol{a})$.

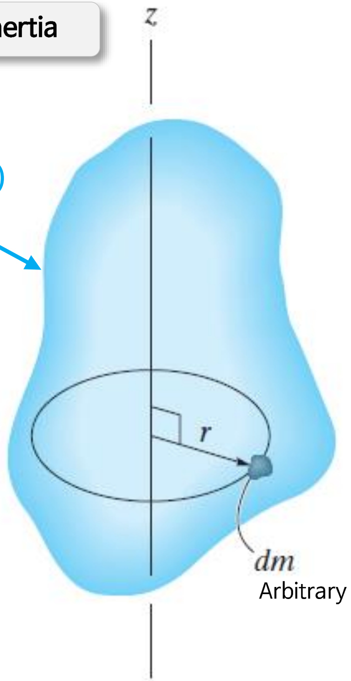
- Moment of inertia is the integral of the "second moment" about an axis of all the elements of mass dm which compose the body.

$$\boldsymbol{I} = \int \boldsymbol{r^2 dm}$$

Moment of Inertia

Moment of Inertia

$\rho = \rho(x, y, z)$



Moment of Inertia with variable Density

- The value of I is different for each axis about which it is computed.
- The moment of inertia about this axis will be denoted as I_G
- The moment of inertia is always a ***positive*** quantity $[kg \cdot m^2]$

- If the body consists of material having a variable density, $\rho = \rho(x, y, z)$, the elemental mass dm of the body can be expressed in terms of its density and volume

$$dm = \rho dV$$

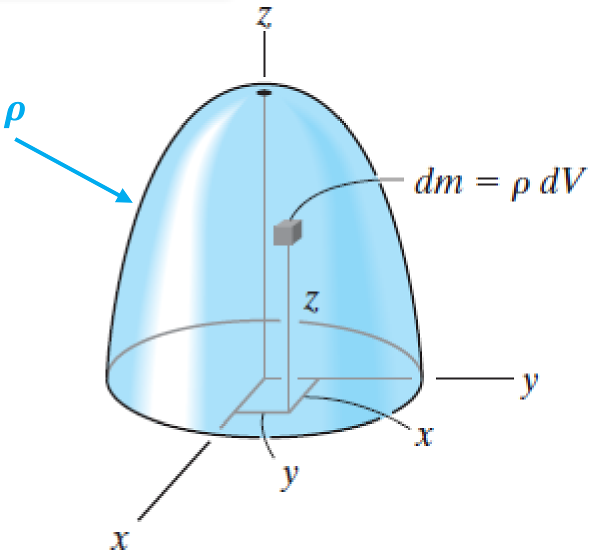
- Then, the body's moment of inertia is computed using volume elements for integration.

$$I = \int r^2 dm$$

$$= \int r^2 \rho dV$$

Moment of Inertia

Moment of Inertia



Moment of Inertia with constant Density

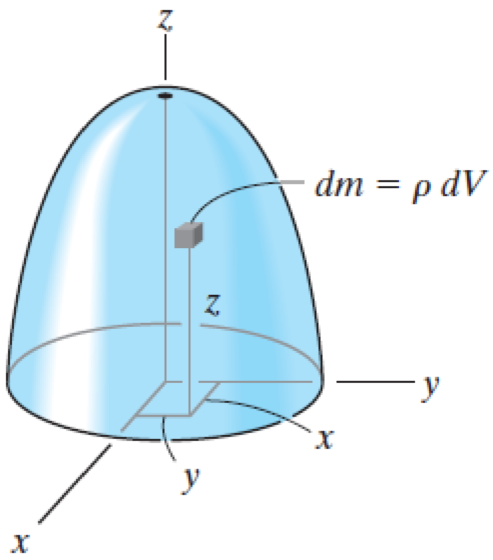
- In the special case of ρ being a constant, this term may be factored out of the integral, and the integration is then purely a function of geometry

$$I = \int r^2 \rho dV = \rho \int r^2 dV$$

- When the volume element chosen for integration has infinitesimal dimensions in all 3-D, the moment of inertia of the body must be determined using "Triple integration"
- The integration process can be simplified to a single integration provided the chosen volume element has a differential size or thickness in only one direction.

Moment of Inertia: Procedure for Analysis

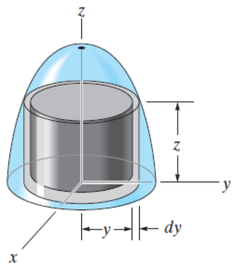
Moment of Inertia



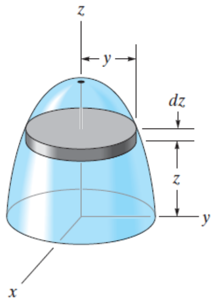
Procedure for Analysis

- To obtain the moment of inertia by integration, we will consider only symmetric bodies having volumes which are generated by revolving a curve about an axis.
- Two types of differential elements can be chosen.

1. Shell Element

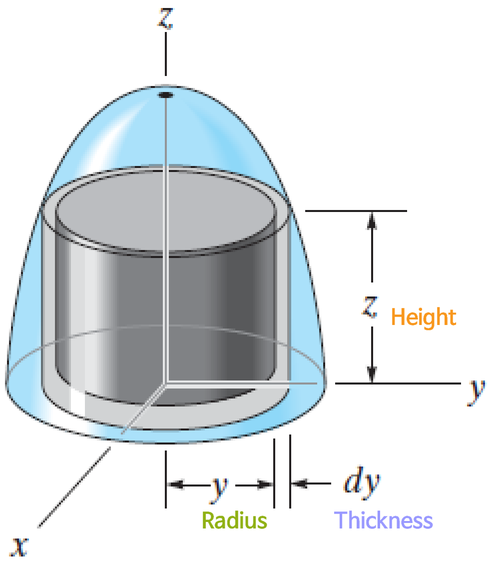


2. Disk Element



Moment of Inertia: Procedure for Analysis

Moment of Inertia



Shell Element

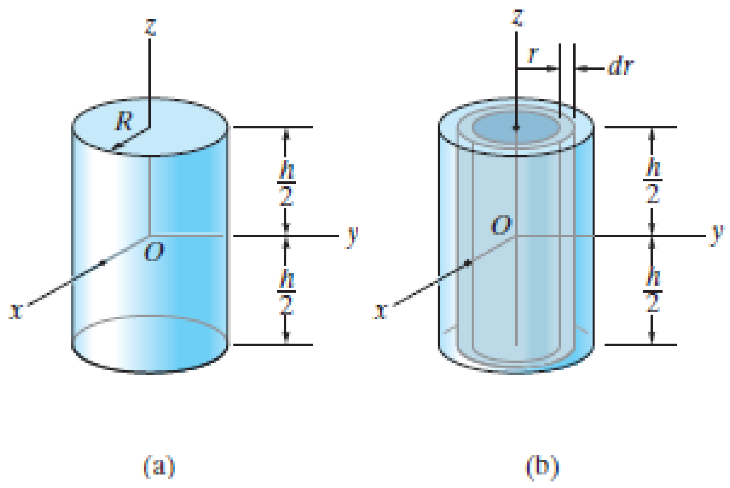
- If a shell element having a height z, radius $r = y$, and thickness dy is chosen for integration, the volume is expressed as

$$dV = (2\pi y)(z)dy$$

- This element may be used for determining the moment of inertia I_z of the body about the z axis
- Entire element, due to its “thickness” lies at the same perpendicular distance $r = y$ from the z axis

Moment of Inertia: Procedure for Analysis

Moment of Inertia



Shell Element

- Using the shell element and a single integration
- The volume of the element is $dV = (2\pi r)(h)dr$, so that its mass is $dm = \rho dV = \rho(2\pi h r dr)$

- The entire element lies at the same distance r from the z axis, the moment of inertia of the element is expressed as

$$dI_z = r^2 dm = 2\rho\pi h r^3 dr$$

- Integrating over the entire region of the cylinder yields

$$I_z = \int r^2 \rho dV = 2\rho\pi h \int_0^R r^3\, dr = \frac{\rho\pi}{2} R^4 h$$

- Considering mass of the cylinder

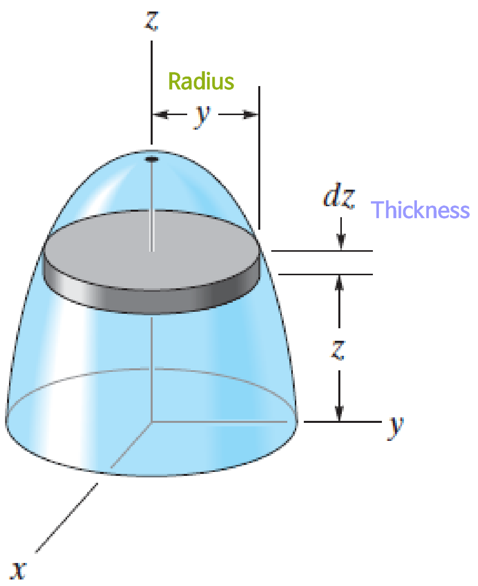
$$m = \int dm = 2\rho\pi h \int_0^R r dr = \rho\pi h R^2$$

- So that

$$I_z = \frac{1}{2} m R^2$$

Moment of Inertia: Procedure for Analysis

Moment of Inertia



Disk Element

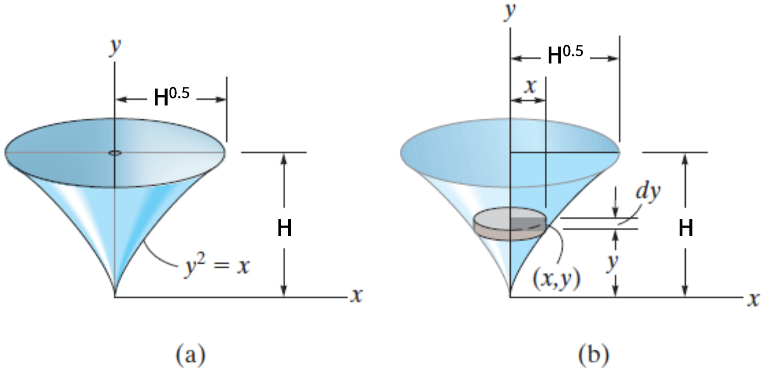
- If a disk element having a radius y and a thickness dz is chosen for integration, the volume is expressed as

$$dV = (\pi y^2)dz$$

- This element is finite in the radial direction, and consequently its parts do not all lie at the same radial distance r from the z axis
- Moment of inertia cannot be determined using $I = \int r^2 \rho dV = \rho \int r^2 dV$ directly
- We should determine the moment of inertia of the element about the z axis and then integrate this results

Moment of Inertia: Procedure for Analysis

Moment of Inertia



Disk Element

- The moment of inertia using a disk element
- The element intersects the curve at the arbitrary point (x, y) and mass is expressed as

$$dm = \rho dV = \rho(\pi x^2)dy$$

- All portions of the element are not located at the same distance from y axis, it is still possible to determine the moment of inertia dI_y of the element about the y axis
- Using $I = \frac{1}{2}mR^2$ from shell element, moment of inertia for the disk element can be expressed as

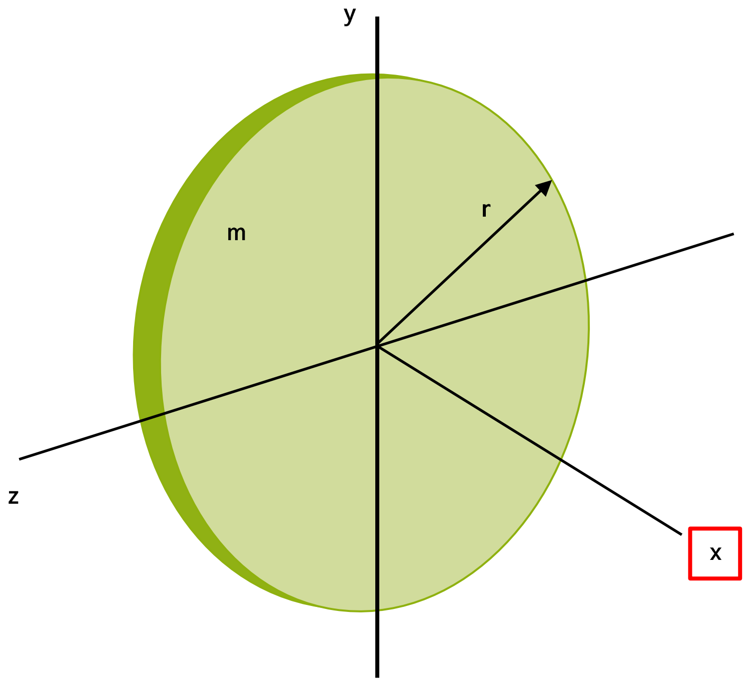
$$dI_y = \frac{1}{2}(dm)x^2 = \frac{1}{2}[\rho(\pi x^2)dy]x^2$$

- Considering $y^2 = x$

$$I_y = \frac{1}{2}\rho\pi\int_0^H x^4 dy = \frac{\rho\pi}{2}\int_0^H y^8\, dy = \frac{1}{18}\rho\pi H^9$$

Moment of Inertia: Quiz I

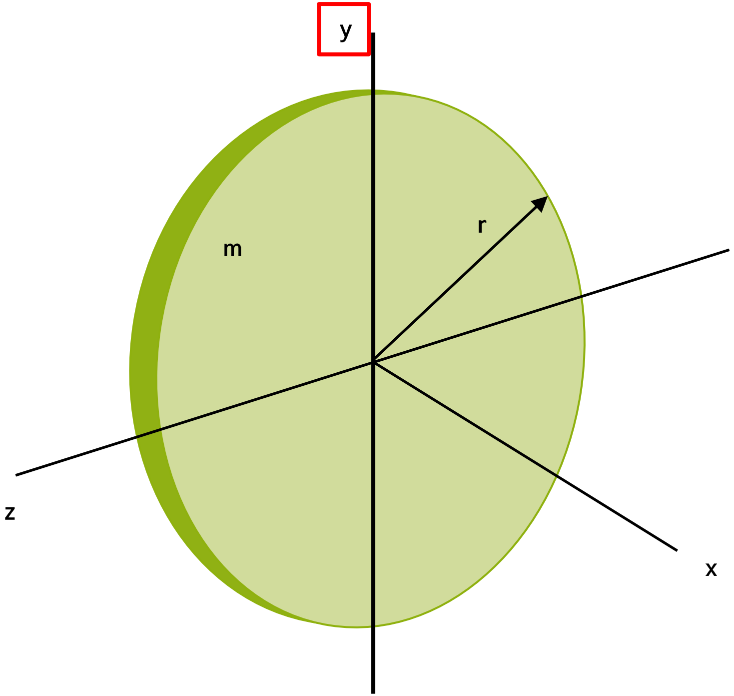
I

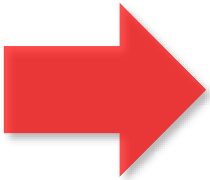


I_x

Moment of Inertia: Quiz I

II

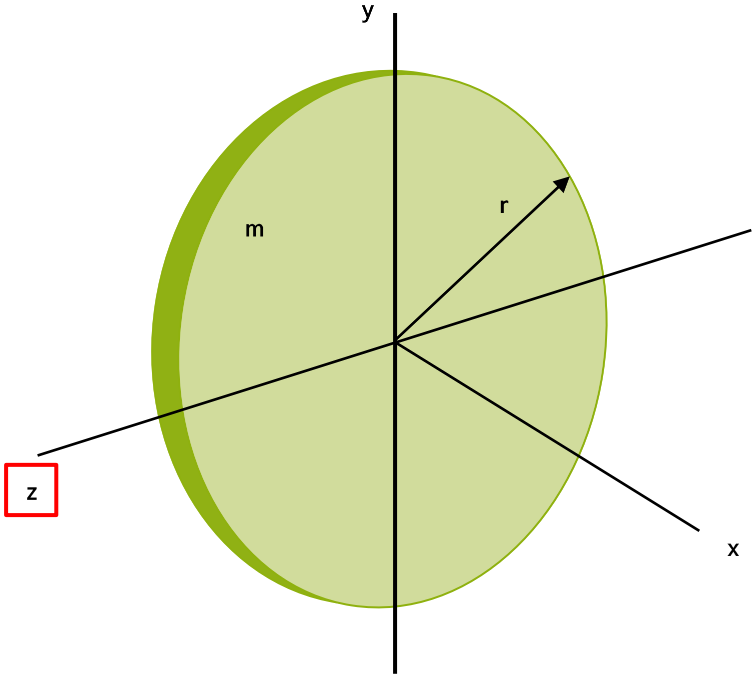




I_y

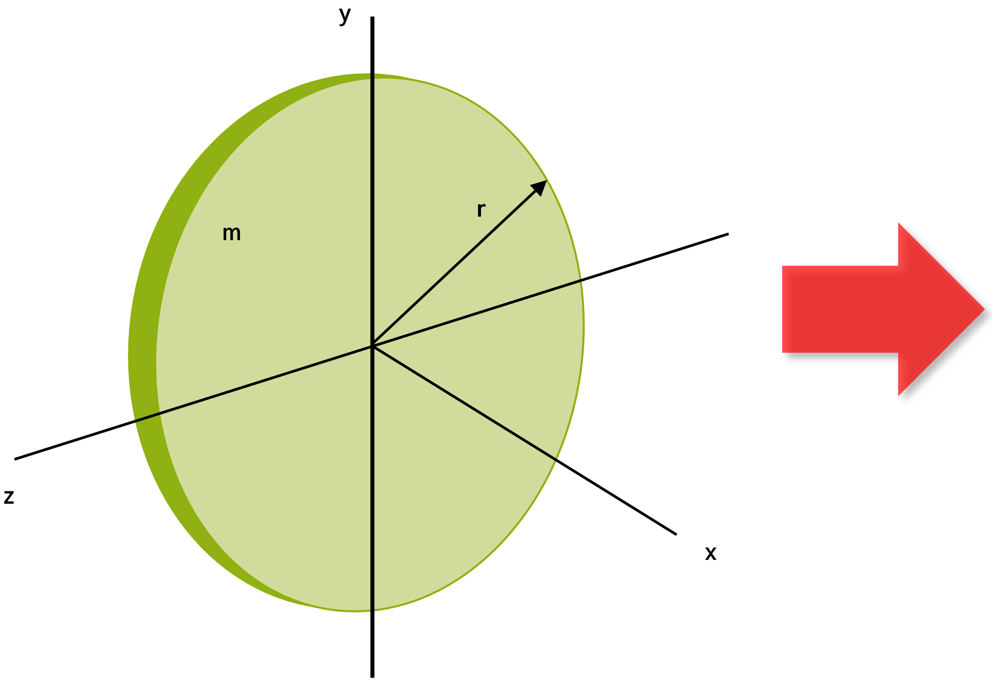
Moment of Inertia: Quiz I

III



I_z

Moment of Inertia: Quiz I



I_x	
I_y	
I_z	

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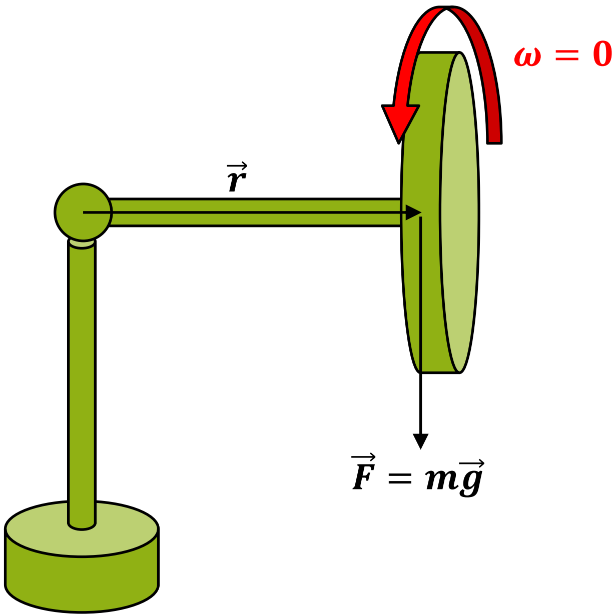
Ⅲ Gyroscopic Motions

Gyroscopic Motions

The torque on a non-spinning gyroscope



$$\vec{\tau} = \vec{r} \times m\vec{g}$$

$$F = ma$$
$$\tau = I\alpha$$

$$mgr = mr^2\alpha$$

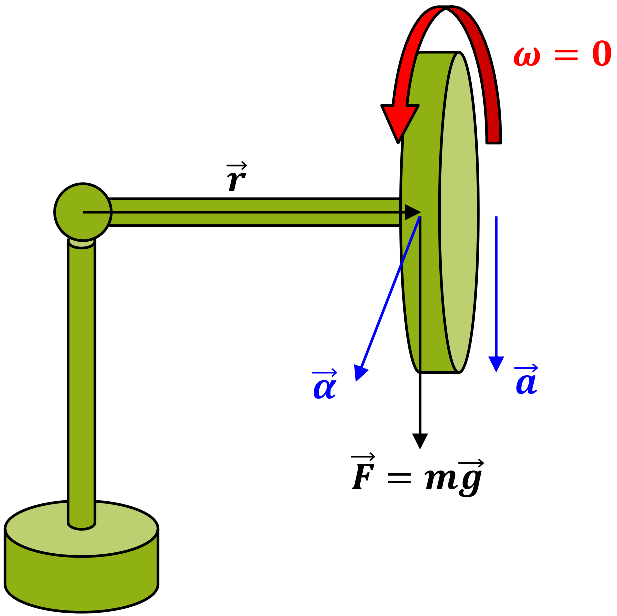
Moment of inertia

$$g = r\alpha$$

$$\alpha = \frac{g}{r}$$

Gyroscopic Motions

The torque on a non-spinning gyroscope



$$\vec{\tau} = \vec{r} \times m\vec{g}$$

$$F = ma$$
$$\tau = I\alpha$$

$$\Rightarrow \quad mgr = mr^2\alpha$$

$$g = r\alpha$$

$$\alpha = \frac{g}{r}$$

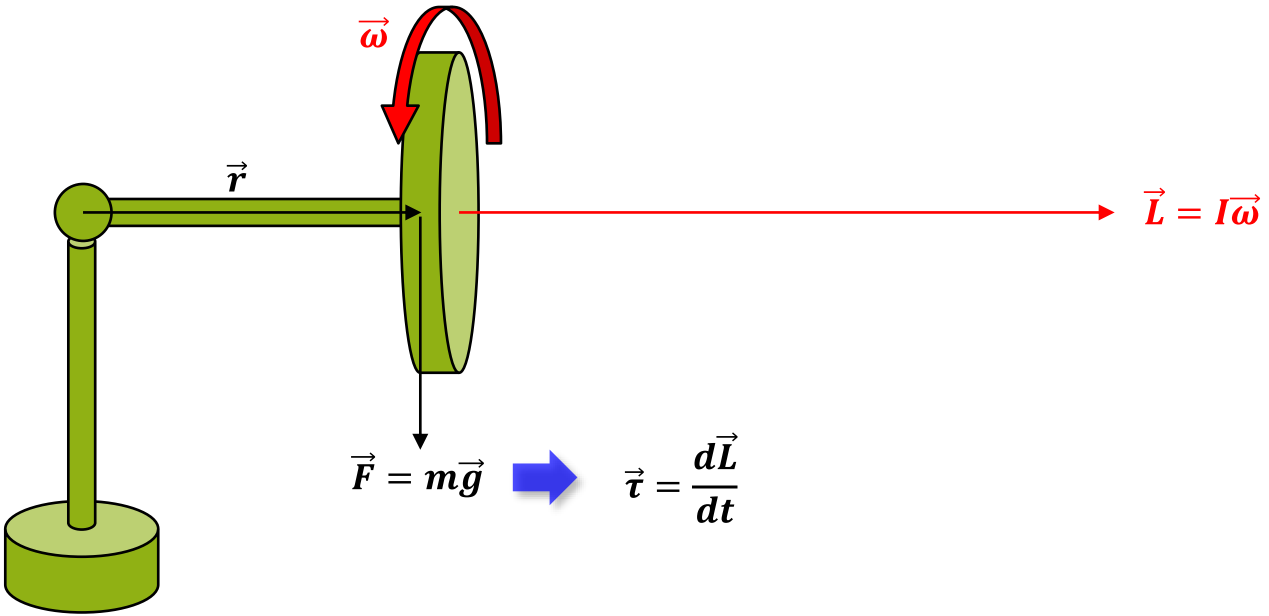
$$a = r\alpha$$

$$\frac{a}{r} = \alpha \quad \text{Angular Acceleration} \quad \Rightarrow \quad \frac{a}{r} = \frac{g}{r}$$

$$a = g$$ Like free-fall

Gyroscopic Motions

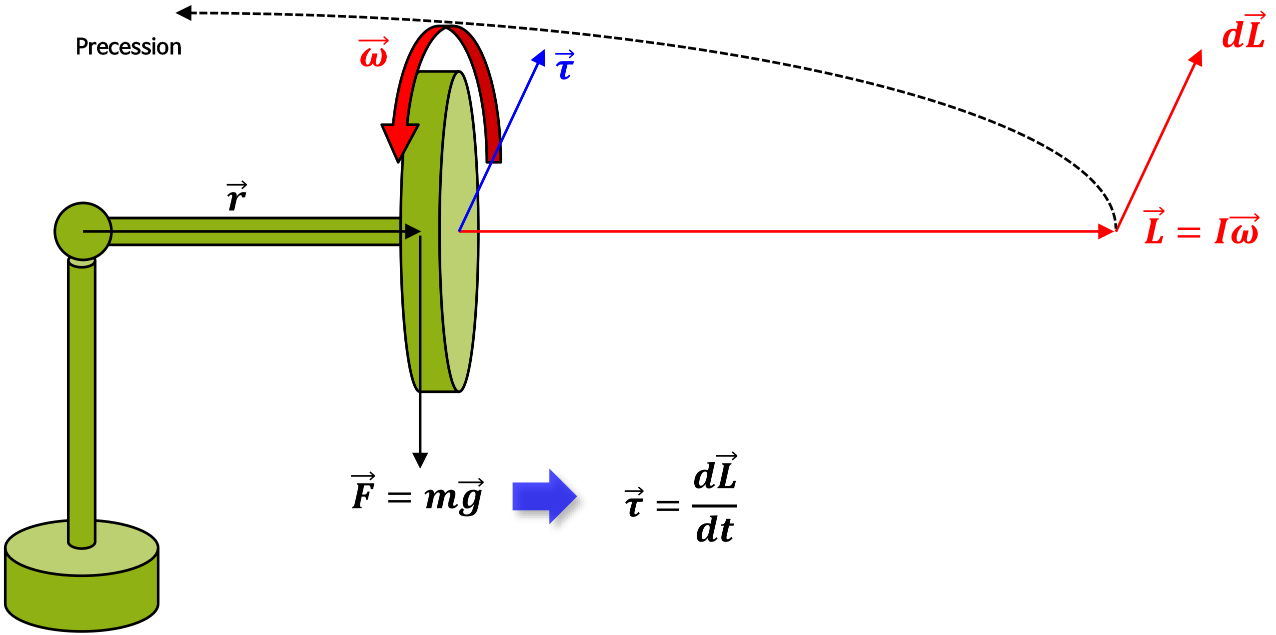
The torque on a spinning gyroscope



$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

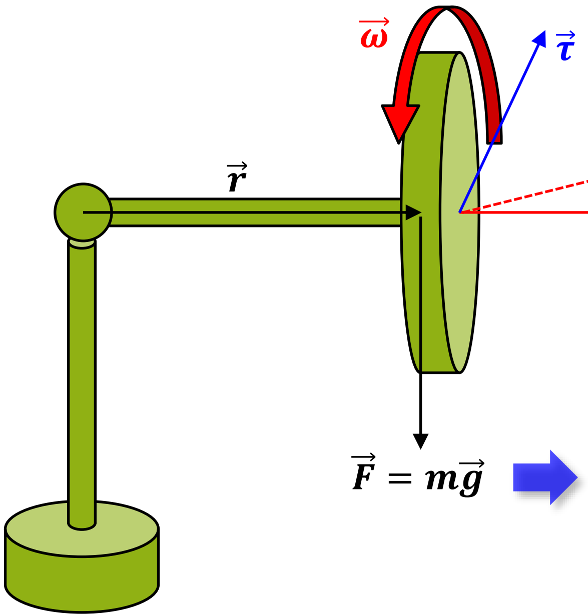
Gyroscopic Motions

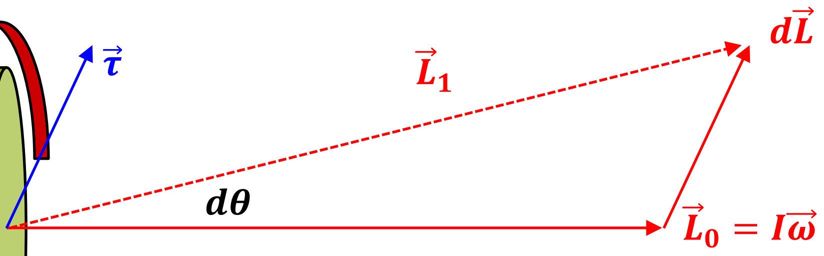
The torque on a spinning gyroscope



Gyroscopic Motions

The torque on a spinning gyroscope





$$\vec{F} = m\vec{g} \Rightarrow \vec{\tau} = \frac{d\vec{L}}{dt}$$

$$\vec{\Omega} = \frac{d\vec{\theta}}{dt}$$

$$= \frac{d\vec{L}}{\vec{L}} \cdot \frac{1}{dt} = \frac{1}{\vec{L}} \cdot \frac{d\vec{L}}{dt}$$

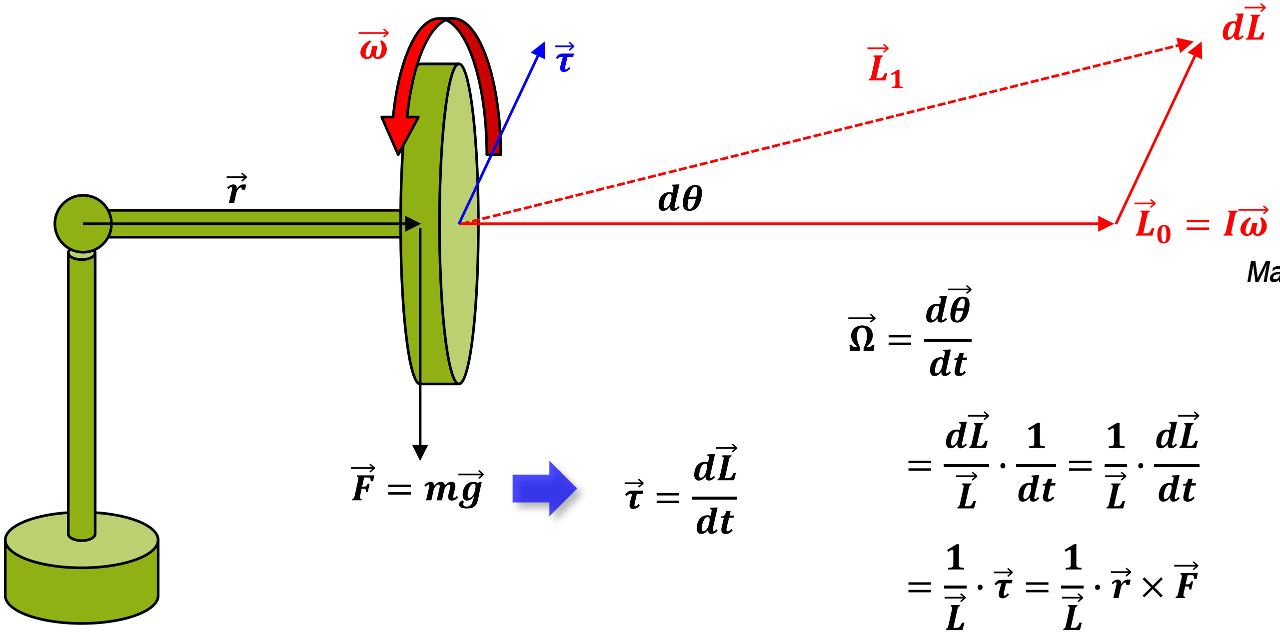
$$= \frac{1}{\vec{L}} \cdot \vec{\tau} = \frac{1}{\vec{L}} \cdot \vec{r} \times \vec{F}$$

$$(\theta \approx sin\theta = tan\theta)$$

$$d\vec{\theta} = \frac{d\vec{L}}{\vec{L}}$$

Gyroscopic Motions

The torque on a spinning gyroscope



$$\vec{F} = m\vec{g} \;\Rightarrow\; \vec{\tau} = \frac{d\vec{L}}{dt}$$

$$\vec{\Omega} = \frac{d\vec{\theta}}{dt}$$

$$= \frac{d\vec{L}}{\vec{L}} \cdot \frac{1}{dt} = \frac{1}{\vec{L}} \cdot \frac{d\vec{L}}{dt}$$

$$= \frac{1}{\vec{L}} \cdot \vec{\tau} = \frac{1}{\vec{L}} \cdot \vec{r} \times \vec{F}$$

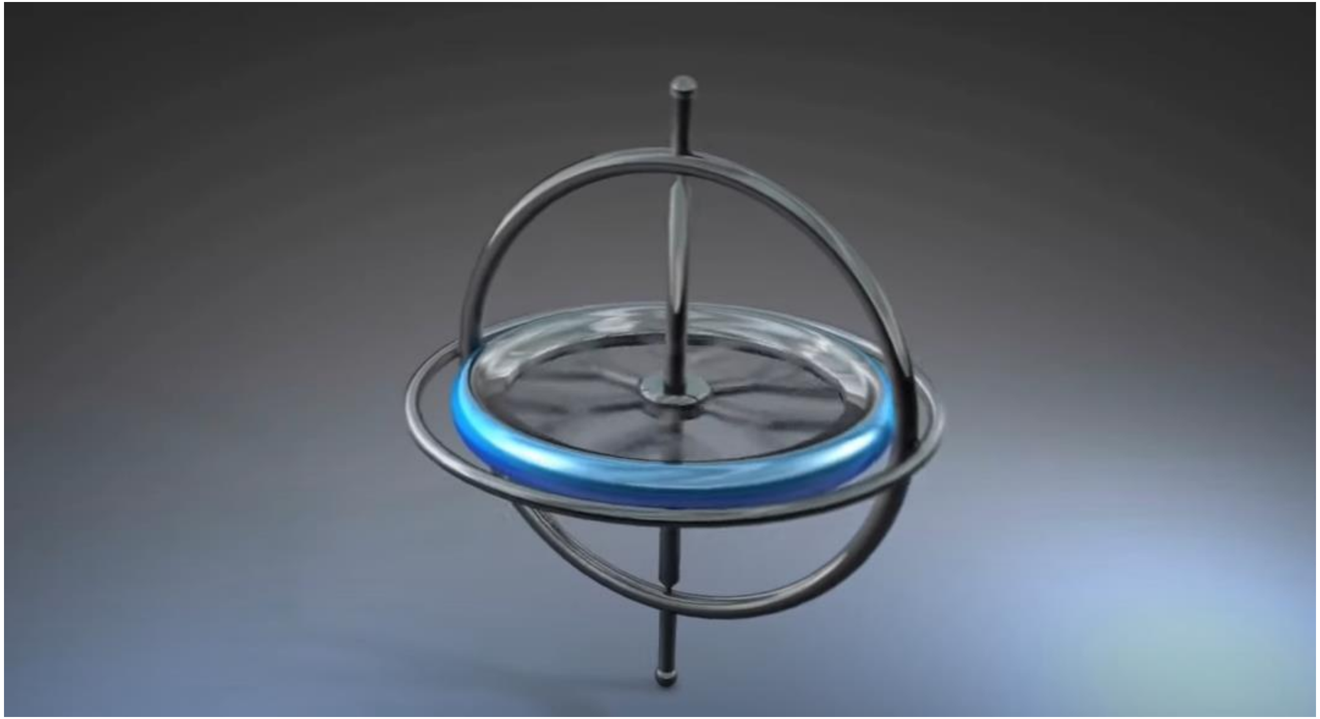
Magnitude of Precession Speed

$$\Omega = \frac{1}{L} rF$$

$$= \frac{1}{mr^2\omega} rmg$$

$$\therefore\ \Omega = \frac{g}{r\omega}$$

Gyroscopic Motions



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