Dynamics COE2004

Gyroscope Ⅰ

: Two-Dimensional Kinetics

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HANYANG UNIVERSITY

Curriculum

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4	12/5	Three-Dimensional Kinetics of a Rigid Body

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Ⅱ Vector Analysis

Ⅲ Translational & Rotational Kinetics in Two-Dimensional Space

Ⅳ Gyroscope

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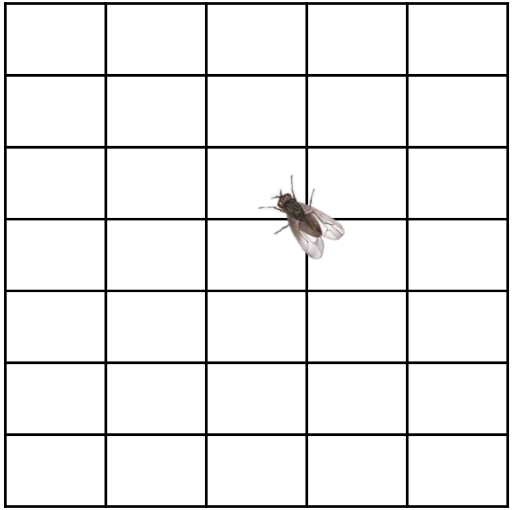
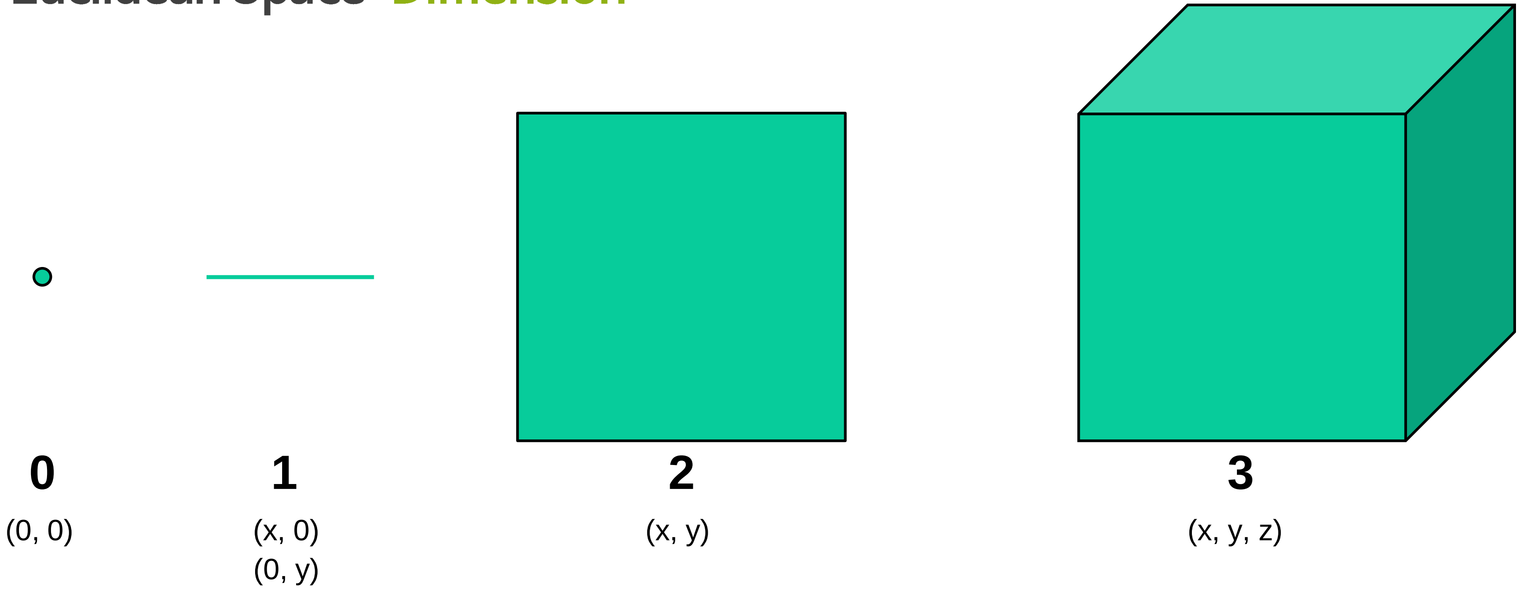
I Introduction

Coordinate Systems: Cartesian Coordinate



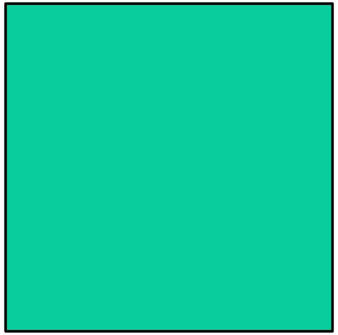
- A cartesian coordinate system developed by Rene Descartes
- A cartesian coordinate for N-dimensional space

Euclidean Space: Dimension



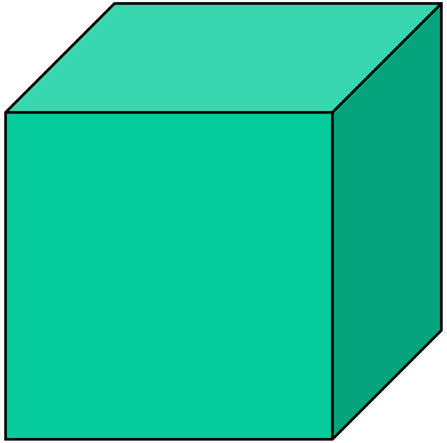
- Number of directions move independently
- Numbers of basis

Euclidean Space: Dimension



2

(x, y)

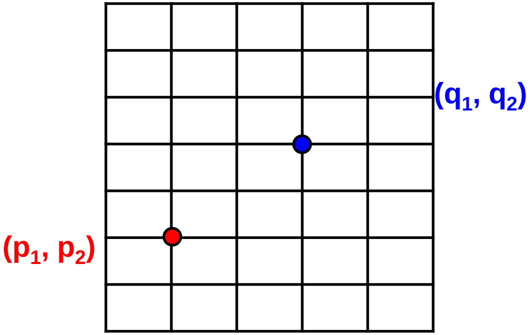


3

(x, y, z)

- Two-dimensional space with two basis
- Three-dimensional space with three basis

Euclidean Space: Euclidean Space



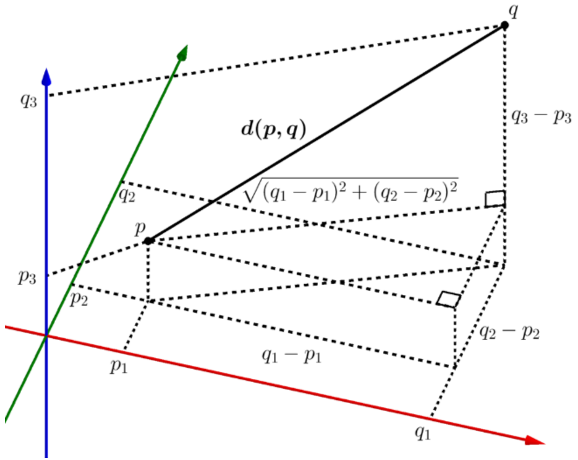
2-D Space

Distance

$$R = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2}$$

- Distance between two particles in two-dimensional space
- $R = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2}$

Euclidean Space: Euclidean Space



3-D Space

Distance

$$R = \sqrt{(q_1-p_1)^2+(q_2-p_2)^2+(q_3-p_3)^2}$$

$$R^n = \sqrt{\sum_{i=1}^{n}(q_n-p_n)^2}$$

- Distance between two particles in three-dimensional space
- $R = \sqrt{(q_1-p_1)^2+(q_2-p_2)^2+(q_3-p_3)^2}$
- $R^n = \sqrt{(q_1-p_1)^2+(q_2-p_2)^2+(q_3-p_3)^2+\cdots+(q_n-p_n)^2}$ for n-dimensional space: Euclidean space

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II Vector Analysis

Vector Analysis: Scalar and Vector

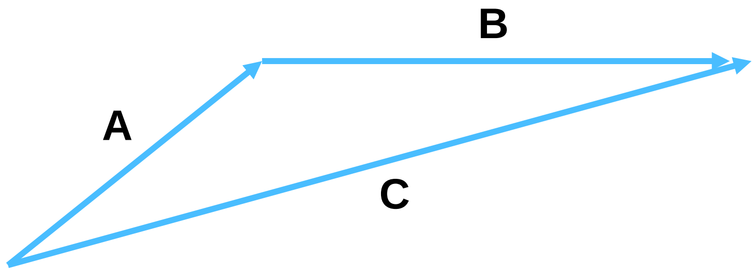
Scalar

- Physical quantity with magnitude only
- Independent with coordinate axis
- Ex) Mass, Time, Temperature

Vector

- Physical quantity with magnitude and associated direction
- Their length and an angle between any vectors remain unaffected by the orientation of coordinates we choose

Vector Analysis: Vector Summations



$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$ Commutative

$$(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$$ Associative

$$\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$$

- Vector summation is commutative
- Vector summation is associative
- Vector subtraction is addition of negative vector (a vector of the same magnitude but with reversed direction)

Vector Analysis: Vector and Vector Space

$$\begin{cases} \vec{x} = (x_1, x_2, x_3) \\ \vec{y} = (y_1, y_2, y_3) \end{cases}$$

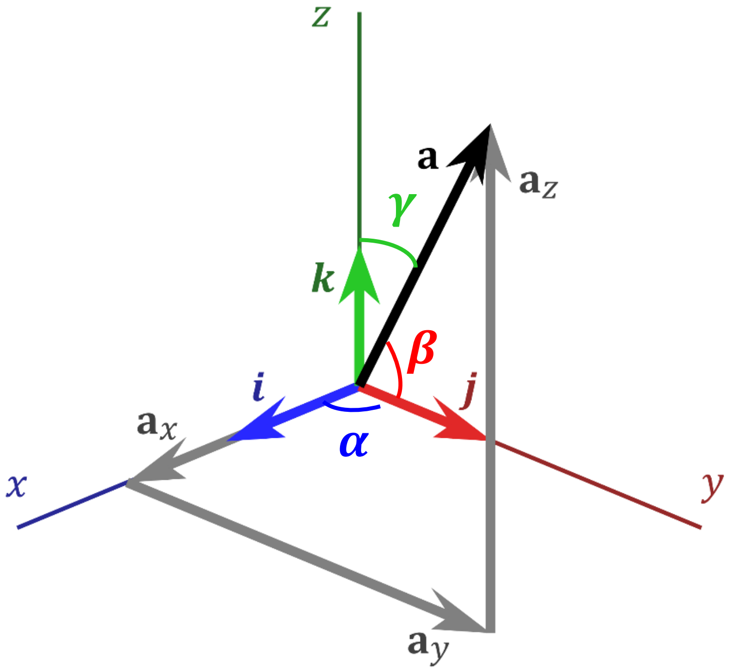
Vector

1. Same vectors
$$\vec{x} = \vec{y} : x_i = y_i$$
2. Vector summations
$$\vec{x} + \vec{y} + \vec{z} : x_i + y_i + z_i$$
3. Scalar multiplications
$$a \cdot \vec{x} = (ax_1, ax_2, ax_3)$$
4. Negative vectors
$$-\vec{x} = (-1) \cdot \vec{x} = (-x_1, -x_2, -x_3)$$
5. Zero vector
$$\mathbf{0} = (0, 0, 0)$$

Vector Space

1. Commutative vector summations
$$\vec{x} + \vec{y} = \vec{y} + \vec{x}$$
2. Associative of vector summations
$$(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$$
3. Distributive of scalar multiplications
$$a \cdot (\vec{x} + \vec{y}) = a\vec{x} + a\vec{y}$$
4. Associative of scalar multiplications
$$(ab)\vec{x} = a(b\vec{x})$$

Vector Analysis: Vector and Vector Space



$$x = a \cdot \cos\alpha$$
$$y = a \cdot \cos\beta$$
$$z = a \cdot \cos\gamma$$

Direction Cosines

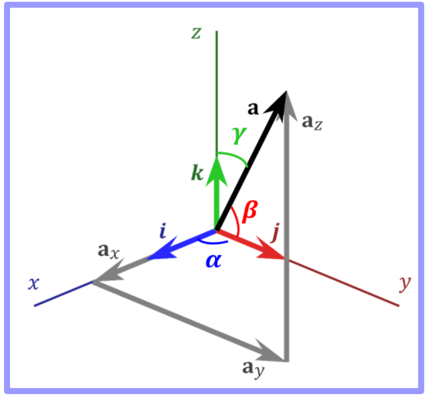
$$\cos\alpha, \cos\beta, \cos\gamma$$

Unit Vectors

$$\hat{a} = \frac{\vec{a}}{a}$$

$\hat{x}$ $(or\ \hat{\imath})$: A vector of unit magnitude pointing in the positive x-dir.
$\hat{y}$ $(or\ \hat{\jmath})$: A vector of unit magnitude pointing in the positive y-dir.
$\hat{z}$ $(or\ \hat{k})$: A vector of unit magnitude pointing in the positive z-dir.

Vector Analysis: Scalar (Dot) Product



Scalar (Dot) Product

- A projection of a vector $\vec{A}$ onto a coordinate axis

$$\vec{A} \cdot \vec{B}$$

$$A_x = A\cos\alpha \equiv \vec{A} \cdot \hat{x}$$
$$A_y = A\cos\beta \equiv \vec{A} \cdot \hat{y}$$
$$A_z = A\cos\gamma \equiv \vec{A} \cdot \hat{z}$$

Note

$\hat{x} \cdot \hat{x} = \hat{y} \cdot \hat{y} = \hat{z} \cdot \hat{z} = 1$: Same directions

$\hat{x} \cdot \hat{y} = \hat{x} \cdot \hat{z} = \hat{y} \cdot \hat{z} = 0$: Different directions

- Results of scalar (dot) product: Scalar

Vector Analysis: Scalar (Dot) Product

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$$
$$\vec{B} = B_x\hat{x} + B_y\hat{y} + B_z\hat{z}$$

$$\vec{A}\cdot\vec{B} = \left(A_x\hat{x} + A_y\hat{y} + A_z\hat{z}\right)\cdot\left(B_x\hat{x} + B_y\hat{y} + B_z\hat{z}\right)$$
$$= A_xB_x + A_yB_y + A_zB_z$$

If θ is an angle between $\vec{A}$ and $\vec{B}$

$$\vec{A}\cdot\vec{B} = AB\cos\theta$$

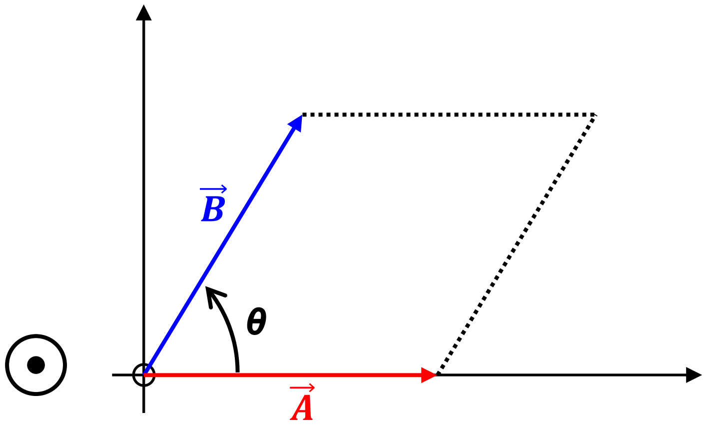
$$\vec{A}\cdot\vec{B} = 0$$

- $\vec{A}$ and $\vec{B}$ must be Perpendicular
- $\vec{A}$ and $\vec{B}$ are Orthogonal

Vector Analysis: Vector (Cross) Product

Vector (Cross) Product

- A perpendicular directions to a plane of $\vec{A}$ and $\vec{B}$



$$\vec{A} \times \vec{B}$$

Cross Product

$\vec{C} = \vec{A} \times \vec{B}$: Result is Vector

Magnitude

$$|\vec{C}| = |\vec{A}||\vec{B}|sin\theta$$

Direction

Perpendicular to the plane of $\vec{A}$ and $\vec{B}$ such that,
$\vec{A}$, $\vec{B}$, and $\vec{C}$ form a right-handed system

Vector Analysis: Vector (Cross) Product

$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$ Not commutative but **anti-commutative**

Note

$\hat{x}\times\hat{x} = \hat{y}\times\hat{y} = \hat{z}\times\hat{z} = 0$

$\hat{x}\times\hat{y} = \hat{z}$ $\hat{y}\times\hat{z} = \hat{x}$ $\hat{z}\times\hat{x} = \hat{y}$

$\hat{y}\times\hat{x} = -\hat{z}$ $\hat{z}\times\hat{y} = -\hat{x}$ $\hat{x}\times\hat{z} = -\hat{y}$

With 3x3 matrix

$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$

$\vec{B} = B_x\hat{x} + B_y\hat{y} + B_z\hat{z}$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\begin{aligned}\vec{A}\times\vec{B} &= \left(A_x\hat{x} + A_y\hat{y} + A_z\hat{z}\right)\times\left(B_x\hat{x} + B_y\hat{y} + B_z\hat{z}\right)\\ &= A_xB_y\hat{z} - A_xB_z\hat{y} - A_yB_x\hat{z} + A_yB_z\hat{x} + A_zB_x\hat{y} - A_zB_y\hat{x}\\ &= \left(A_yB_z - A_zB_y\right)\hat{x} + \left(A_zB_x - A_xB_z\right)\hat{y} + \left(A_xB_y - A_yB_x\right)\hat{z}\end{aligned}$$

Vector Analysis: Vector (Cross) Product

I About $\hat{x}$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

(+) (-)

$\vec{A}\times\vec{B}$ ➡ $A_yB_z\hat{x} - A_zB_y\hat{x}$

Vector Analysis: Vector (Cross) Product

II About $\hat{y}$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

(-) (+)

$\vec{A} \times \vec{B}$ ➡ $A_y B_z \hat{x} - A_z B_y \hat{x} + A_z B_x \hat{y} - A_x B_z \hat{y}$

Vector Analysis: Vector (Cross) Product

III About $\hat{z}$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

(+) (-)

$\vec{A} \times \vec{B}$ ➡ $A_y B_z \hat{x} - A_z B_y \hat{x} + A_z B_x \hat{y} - A_x B_z \hat{y} + A_x B_y \hat{z} - A_y B_x \hat{z}$

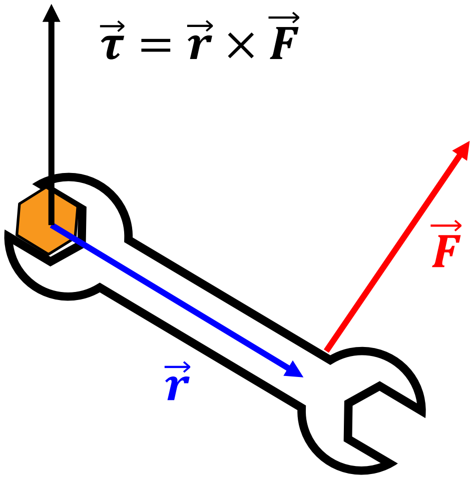
$= (A_y B_z - A_z B_y)\hat{x} + (A_z B_x - A_x B_z)\hat{y} + (A_x B_y - A_y B_x)\hat{z}$

Vector Analysis: Vector(Cross) Product

Physical quantity of vector(cross) product

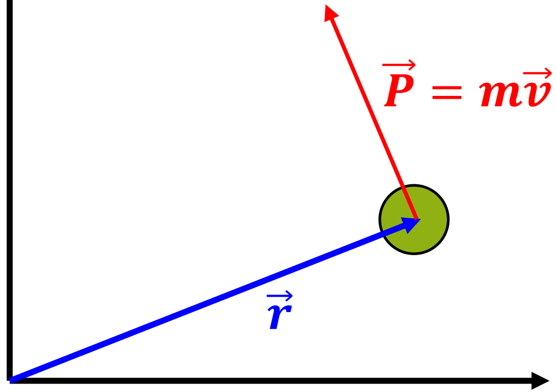
Torque

$$\vec{\tau} = \vec{r} \times \vec{F}$$



Angular Momentum

$$\vec{L} = \vec{r} \times \vec{P}$$



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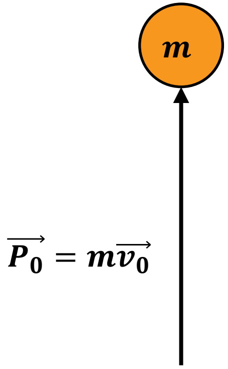
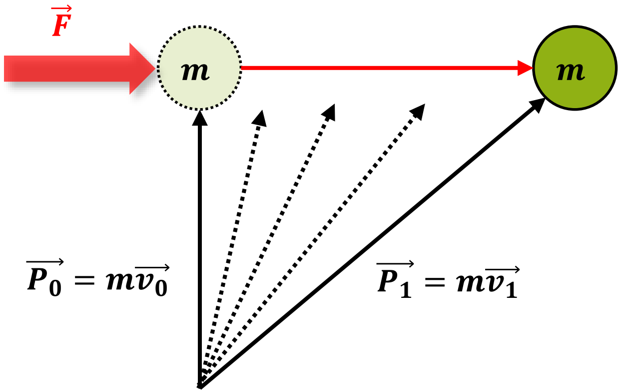
III Translational & Rotational Kinetics in Two-Dimensional Space

Translational Kinetics in Two-Dimensional Space

A moving ball ($f = 0$)

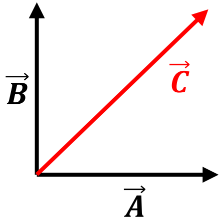


A forced ball ($f = 0$)



Vector summations

$$\vec{A} + \vec{B} = \vec{C}$$



$$\vec{P_0} + \vec{F} = \vec{P_1}$$

$$\Rightarrow \vec{F} = \vec{P_1} - \vec{P_0} = m\vec{v_1} - m\vec{v_0}$$

Derivative about time

$$\vec{F} = \Delta m\vec{v_n}$$

$$= \frac{d}{dt}(m\vec{v_n})$$

$$= \vec{v_n}\frac{dm}{dt} + m\frac{d\vec{v_n}}{dt}$$

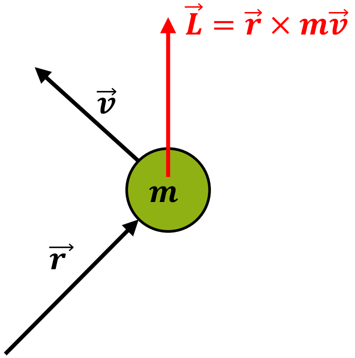
$$\Rightarrow \vec{F} = m\vec{a}$$

$$\vec{F} = \frac{d\vec{P}}{dt}$$

Force [kg·m/s^2]
: Momentum [kg·m/s] change about unit time [s]

Rotational Kinetics in Two-Dimensional Space

Angular momentum



Derivative about time

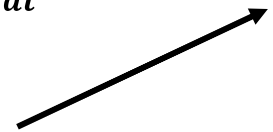
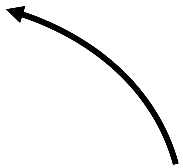
$$\frac{d\vec{L}}{dt} = \frac{d(\vec{r} \times m\vec{v})}{dt}$$

$$\frac{d\vec{L}}{dt} = \vec{v} \times m\vec{v} + \vec{r} \times m\vec{a} = \vec{r} \times m\vec{a} = \vec{r} \times \vec{F}$$

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

Torque [kg·m^2/s^2]
: Angular momentum [kg·m^2/s] change about unit time [s]

Translational & Rotational Kinetics in Two-Dimensional Space

Translational kinetics	Rotational kinetics
Linear momentum $\vec{P}$	Angular momentum $\vec{L}$
Force $\vec{F}$	Torque $\vec{\tau}$
$\vec{F} = \frac{d\vec{P}}{dt}$ 	 $\vec{\tau} = \frac{d\vec{L}}{dt}$

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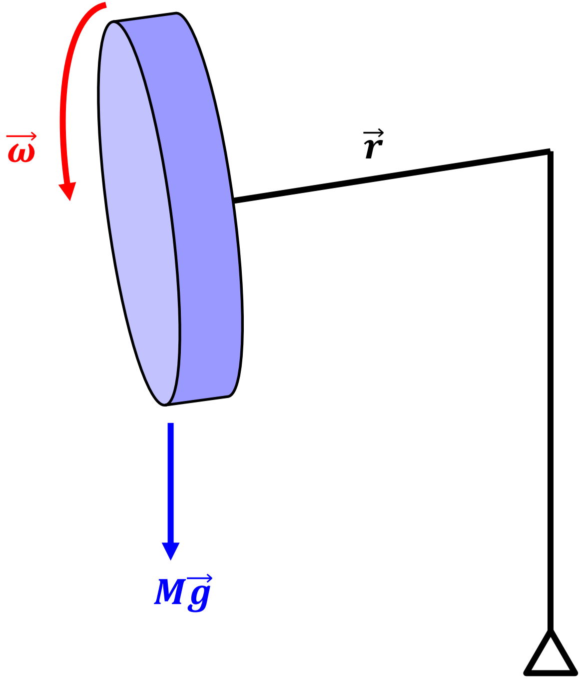
Ⅳ Gyroscope

Gyroscope

Gyroscope

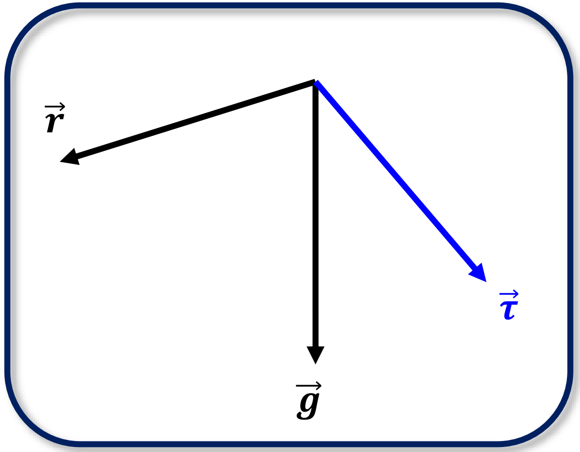
M [kg]
I [kgm^2]

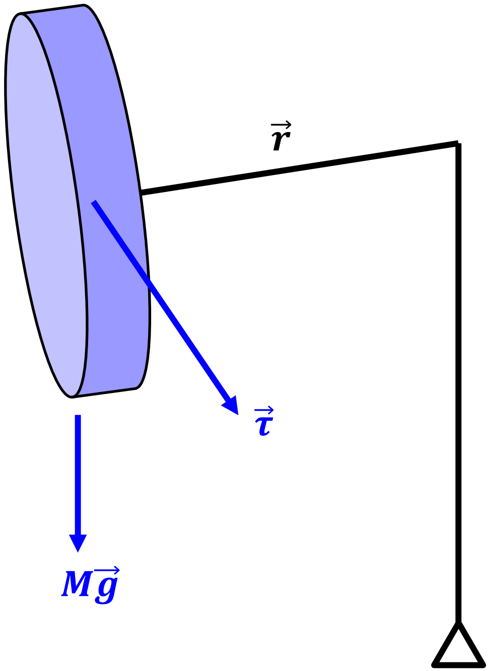


Gyroscope

Torque

M [kg]
I [kgm^2]

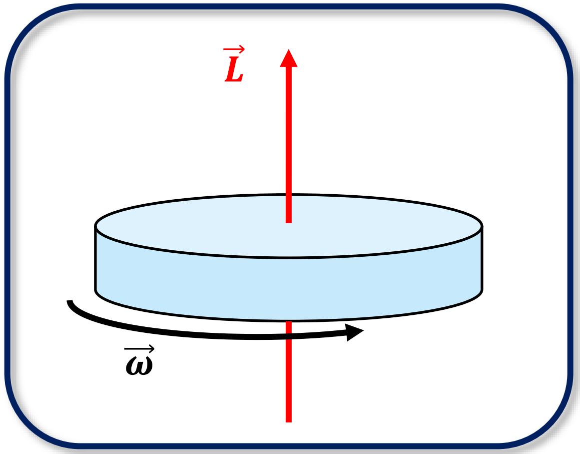


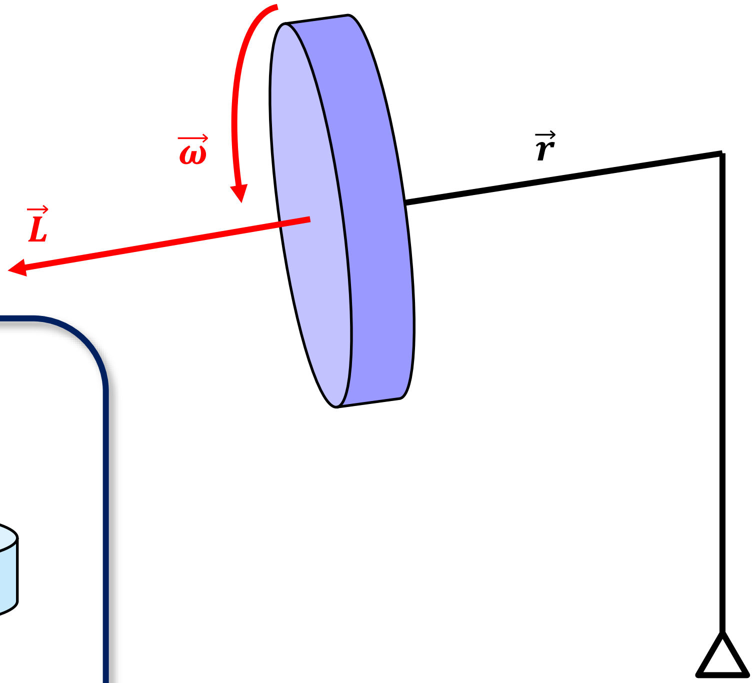


Gyroscope

Angular momentum

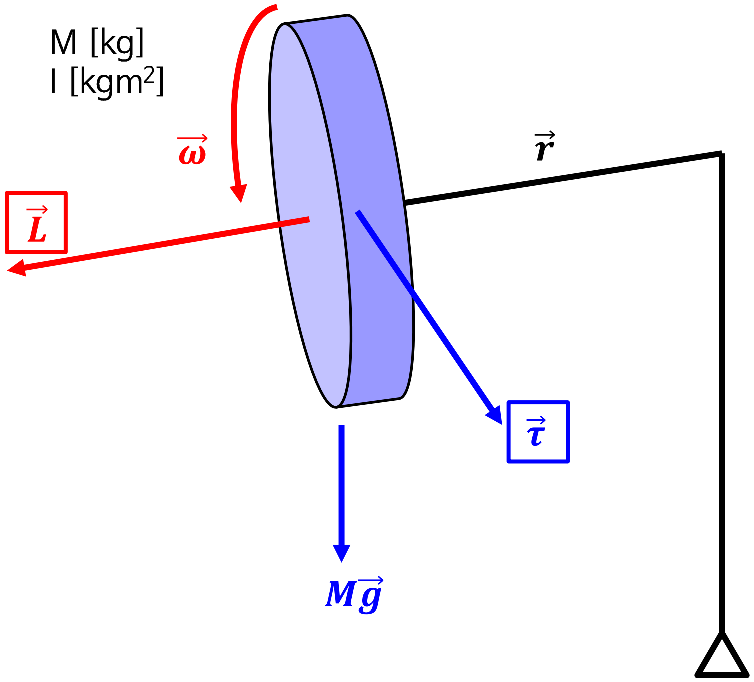
M [kg]
I [kgm^2]





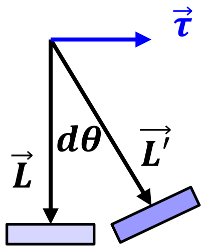
Gyroscope

M [kg]
I [kgm²]



$$\vec{\tau} = \frac{d\vec{L}}{dt} = \frac{d(I\vec{\omega})}{dt} = I\vec{\alpha}$$

$$\vec{\tau}dt = d\vec{L}$$



$$d\vec{L} = Ld\theta$$

$$\frac{d(L\theta)}{dt} = \vec{\tau} = L\frac{d\theta}{dt} = L\Omega$$

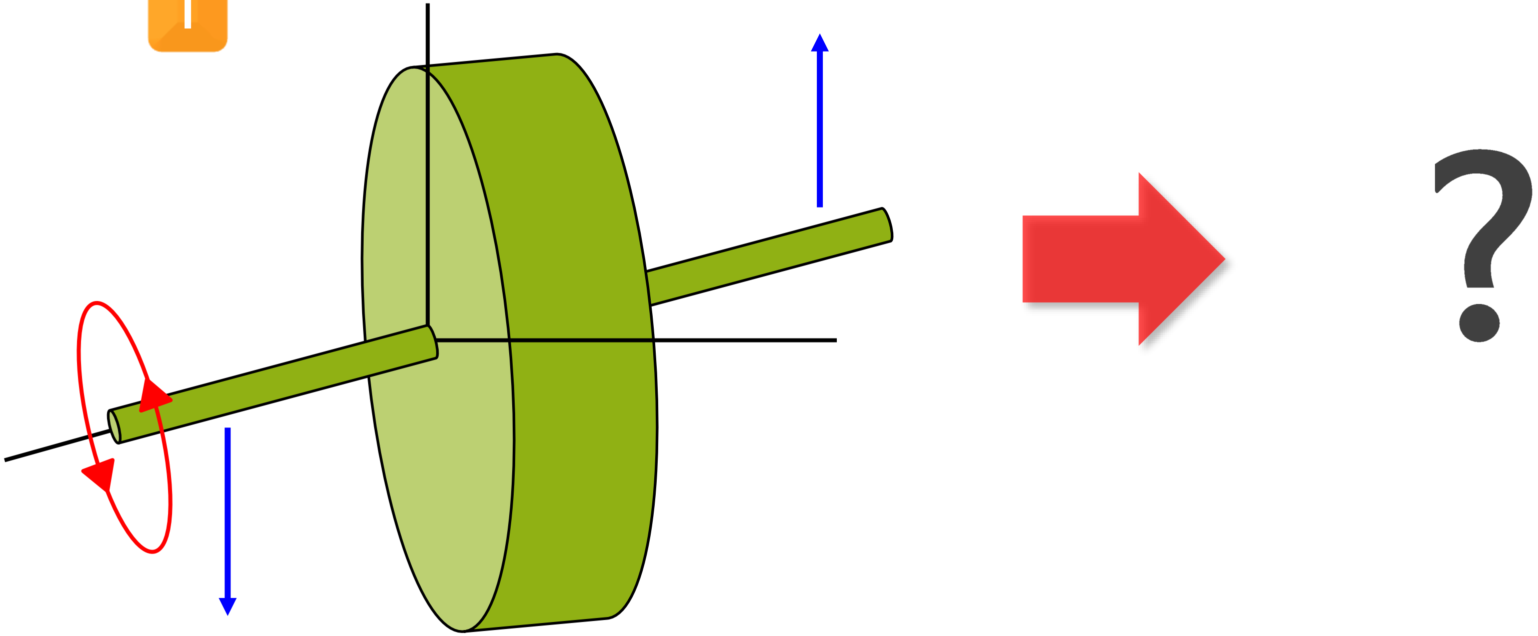
Magnitude

$$rMg = L\Omega = I\omega\Omega$$

$$\therefore \Omega = \frac{rMg}{I\omega}\ [rad/s]$$

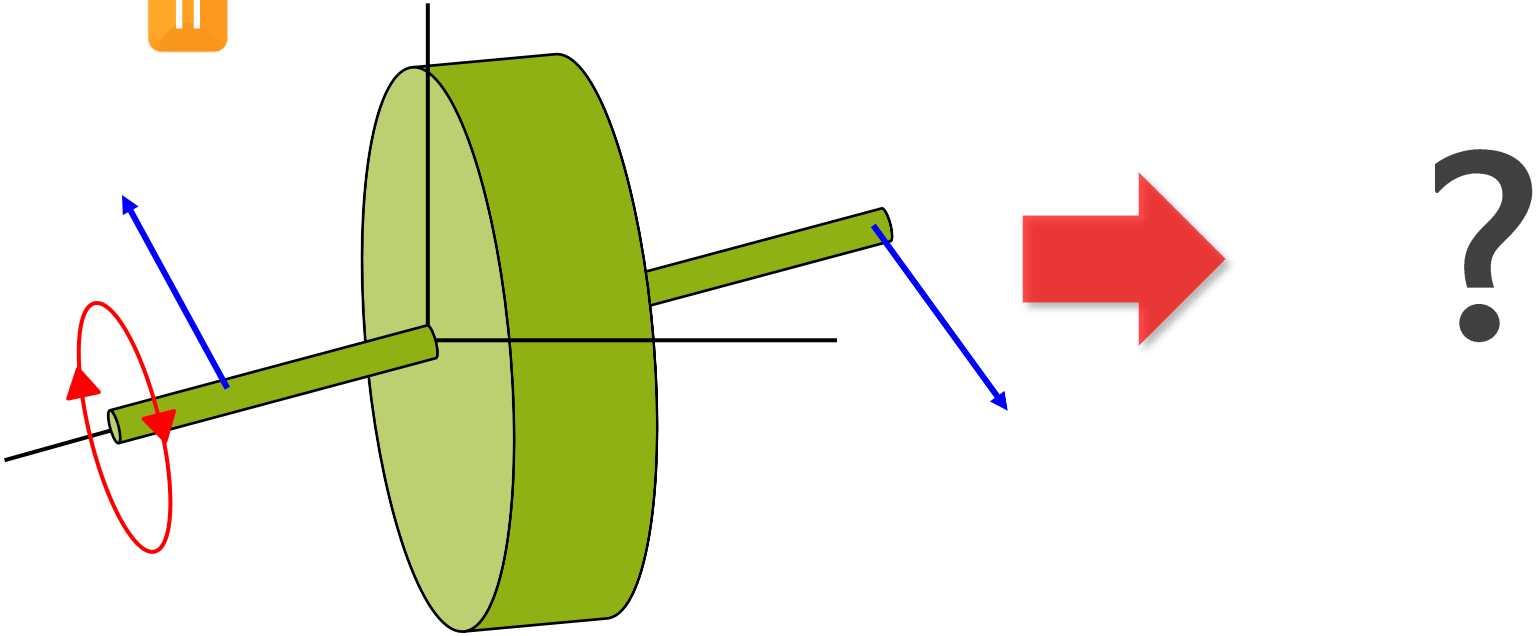
Gyroscope: Examples

I



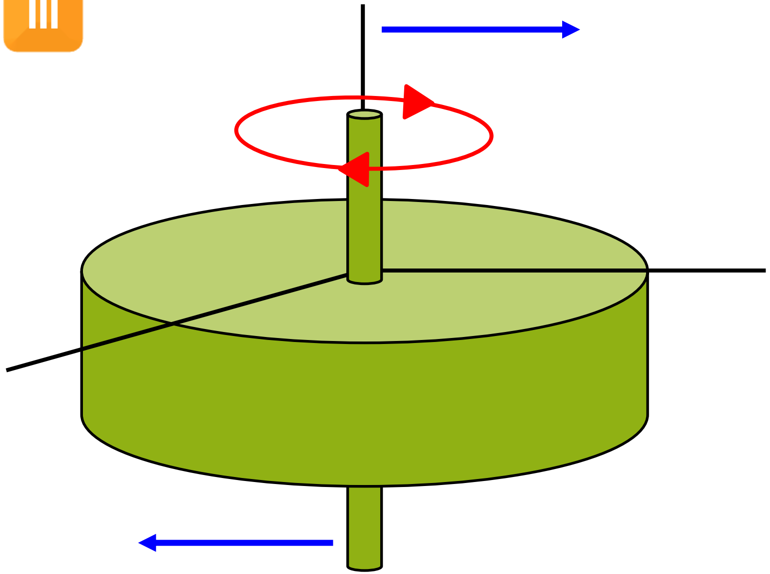
Gyroscope: Examples

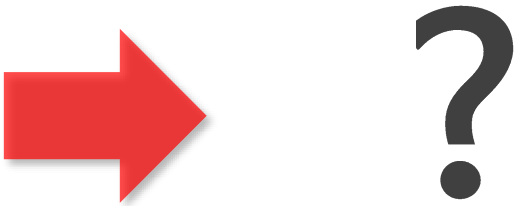
II



Gyroscope: Examples

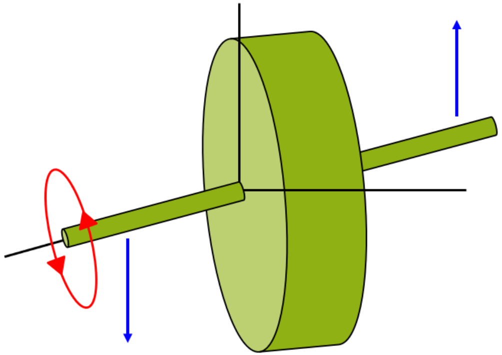
III



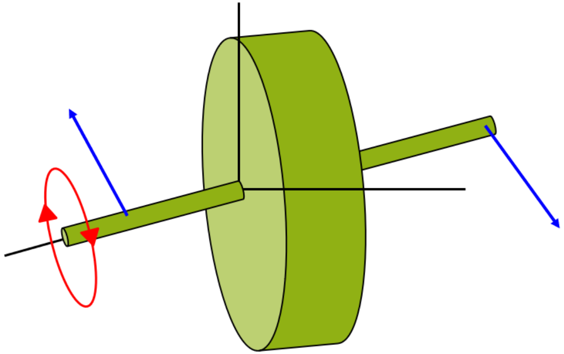


Gyroscope: Examples

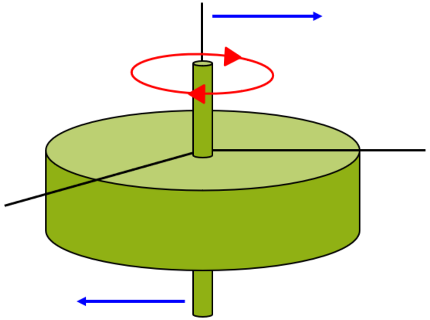
I



II



III

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